AD

30346

LL AERONAUTICAL LABORATORY, INC.

Report No. AF-814-A-2

DETERMINATION OF THE EXTERNAL CONTOUR OF A BODY OF REVOLUTION WITH A CENTRAL DUCT SO AS TO GIVE MINIMUM DRAG IN SUPERSONIC FLOW, WITH VARIOUS PERIMETRAL CONDITIONS IMPOSED UPON THE MISSILE GEOMETRY

Part III - Numerical Application

by

Carlo Ferrari

November 1953

Contract No. N6ori-119, T.O. IV

#1

Corrected Copy

B U F F A L O , N E W Y O R K

An error was found in the application of the formula providing the L and L' adjusting terms in the second approximation stage, as carried out in the previously issued report. This mistake affects most of the subsequent results to a small degree, but in no case are the actual duct coordinates changed by as much as one percent. The corrections have been meticulously made, however, beginning on page 48, in order not to detract from the educational value of the report, but nothing of any consequence in regard to design can be attributed to the slight numerical changes which have resulted from these recomputations.

PLEASE DESTROY THE PREVIOUSLY ISSUED UNCORRECTED COPY OF THIS REPORT

CORNELL AERONAUTICAL LABORATORY, INC.
Buffalo, New York

Correct Copy

Report No. AF-814-A-2
Contract No. N6ori-119,T.O.IV

DETERMINATION OF THE EXTERNAL CONTOUR OF A BODY OF REVOLUTION WITH A CENTRAL DUCT SO AS TO GIVE MINIMUM DRAG IN SUPERSONIC FLOW, WITH VARIOUS PERIMETRAL CONDITIONS IMPOSED UPON THE MISSILE GEOMETRY

Part III - Numerical Application

by
Carlo Ferrari

November 1953

SUMMARY

Several numerical examples have been worked out in detail, by application of a theoretical procedure developed in Part I of this study, in order to determine, consistent with the accuracy inherent in the use of linear theory, what the optimal contour of an annular duct must be if it is to produce minimum external wave drag in supersonic flow, under the stipulations that either (1) the area enclosed between the sought contour and its inner cone-like wall is to have a set constant value, measured in any meridional plane through the duct's axis, or (2) the volume enclosed between the surface of revolution swept out when the sought meridional contour is rotated about its axis and the frustum-of-a-cone inner surface is to be a constant given value.

The specific values that have been selected for the parameters which describe the general features of the free-stream flow and the duct's overall geometry are such as to ensure enough generality so as to cover practically all cases of interest in practical applications. In all the cases examined it is found that the coordinates of the best ducts, which correspond to the various perimetral conditions that are involved, turn out to be scarcely distinguishable from the contour which would be obtained in each specific case by making use of the simple assumption that the pressures created on the duct are directly related to the local slope of the planar elements according to the dictates of Ackeret's two-dimensionally valid formula. This far-reaching conclusion greatly simplifies the problem of the designer in determining such optimum duct contours in practice, but one must not lose sight of the fact that the local pressures obtained by use of the Ackeret formula are <u>not</u> going to be just minor variants from the actual pressures which would be

obtained by use of the truly three-dimensional approach described in the theory now under examination. The amounts that the local pressures on the various ducts differ, between the two- and three-dimensional treatments, are not worked out in all detail, but a comparison of the general harmonic terms, which go to make up the total axial components of the induced velocities at the duct's surface, shows that the terms of small order are significantly different in the two contrasting approaches.

I. INTRODUCTION

The theory developed in Part I of this study[1] is now to be worked out in numerical detail for the general case defined by the values

$$\beta B = 0.25 \text{ and } \ell_1 = 3 \qquad (1)$$

where B is the cotangent of the Mach angle and β is the semi-vertex angle of the frustum of a cone which constitutes the inner surface of the cone-like central duct of the body of revolution being sought. In addition, $\ell_1 = \frac{L}{BR_1}$ represents a "reduced" length for the duct inasmuch as it stands for the axial length of the duct, measured in abscissa units which may be defined as the maximal distance out ahead of the duct, and on its axis, where a small disturbance must be located in order that it should be felt only downstream of the position of the duct's lip at radial distances equal to or larger than the one defining the size of the duct's mouth.

The external contour of the body of revolution having such a central duct is then to be determined in such wise as to give minimum drag under one or more of several perimetral conditions that are further imposed on the geometry of the sought missile shape. The further conditions which are thought to be of some practical consideration are of the following types:

(a) No perimetral condition imposed, except to assume that the external contour actually closes without a blunt base; i.e., it is merely assumed that the "end-points" N_1 and N_2 at the lip and at the tail end of the sought duct contour, respectively, are to belong to the frustum-of-a-cone section which constitutes the inner duct whose large end and small end have radii in the ratio $\frac{R_2}{R_1} = 1 + \beta \cdot \frac{L}{R_1}$; where, of course, R_1 is now equal to the radial coordinate of the N_1 lip-point and R_2 is equal to the radial coordinate of the N_2 tail-point of the external contour. This example is worked out for two specific free-stream flow conditions, defined as

<u>Normalizing Mach No. Case:</u>

$M_\infty = \sqrt{2}$, so $B = \sqrt{M_\infty^2 - 1} = 1$, and thus $\beta = 0.25$ and $L/R_1 = 3$

Small Conical Diffusor Angle Case:

$M_\infty = 1.9436$, so $B = \sqrt{M_\infty^2 - 1} = 1.6667$, and thus $\beta = 0.15$ and $L/R_1 = 5$. (This type of duct might better be designated merely the High Fineness Ratio Case, since the diffusion process taking place in the internal flow is entirely divorced in this study from the flow over the external contours being sought; nevertheless, there is a certain amount of sanction to be found for this usage in the parlance of jet engine designers who are most interested in the available space in the nose even though the customary use of a spike central body will normally decrease the cross-sectional area for the internal flow immediately behind the entrance).

(b) In addition to the closure condition, it is postulated that the area enclosed between the sought contour and the x-axis is to be equal to the area which would be enclosed between this x-axis and the straight line segment joining points N_1 and N_2 (this case will be designated as "No Increase in Area Over that Occupied by Frustum-of-Cone Basic Shape"). Of course, the inner duct will have to have some other shape than the frustum of a cone delimited by the N_1 and N_2 end points, but the flow through the inner duct is not being considered in any of these studies. Same free-stream flow conditions as in (a).

(c) In addition to the closure condition, it is postulated that the area enclosed between the sought contour and the chord line joining the N_1 and N_2 endpoints is to be equal to the area which would be enclosed between this chord line between N_1 and N_2 and lying under the curve with equation

$$\frac{R}{R_1} = 1 + \beta \frac{x}{R_1} + 0.2 \frac{x}{R_1} - 0.2 \frac{x^2}{R_1^2} \cdot \frac{R_1}{L}$$

where the system of Cartesian coordinates employed here in a meridional plane through the body axis is so orientated that the x-axis and body-axis coincide, while the R-axis is taken to pass through the lip end-point, N_1. It may be noted that this parabolic contour, used for sake of comparison, is such that at the mid-point along the duct axis where $x/L = 0.5$ the

"thickness" of the conical duct fairing is 5% of the duct length; i.e., at $x/L = 0.5$ it is found that $\Delta R/L = \dfrac{R-(R_1 + 0.5\ \beta\ L)}{L} = 0.05$.

The free-stream flow conditions are the same as in (a).

(d) In addition to the closure condition, it is postulated that the volume enclosed between the solid of revolution swept out by the sought contour when revolved about the duct axis and the frustum of a cone which has the chordline $N_1 N_2$ as a generatrix shall be equal to the volume enclosed between this same inner truncated-cone type of body of revolution and the one which is swept out as a result of a complete rotation, about the duct's central axis, of the parabolic surface defined according to the equation

$$\frac{R}{R_1} = 1+\beta\,\frac{x}{R_1} + 0.07725\left(\frac{x}{R_1} - \frac{x^2}{R_1^2}\cdot\frac{R_1}{L}\right)$$

in the case of the Normalizing Mach No.: $M_\infty = \sqrt{2}$, or $B = 1$, and $\beta = 0.25$, while the generating curve is similarly taken to be

$$\frac{R}{R_1} = 1+\beta\,\frac{x}{R_1} + 0.16058\left(\frac{x}{R_1} - \frac{x^2}{R_1^2}\cdot\frac{R_1}{L}\right)$$

in the case of the Small Conical Diffusor Angle, when $M_\infty = 1.9436$, $B = 1.6667$, and $\beta = 0.15$

In addition, the parabolic reference meridonal line defined in paragraph (c) has also been used in applying this volume-type of perimetral condition for both the Normalizing Mach No. Case and the Small Conical Diffusor Angle Case.

The procedure to be followed in the numerical working out of these examples is the one discussed in Part I of the theoretical section of this study. Because of the nature of the results found after making the numerical

applications for the examples under consideration according to this first procedure, it is evident that there would be very little point in going on to rederive the optimum contours by use of the second of the two alternative theoretical approaches presented in Part II of the previous work. As a matter of fact, one may make the following sweeping generalization from appraisal of the numerical results calculated in what follows: as far as the determination of the optimal external duct contour is concerned, it will require close observation to detect any difference whatsoever between the best duct shapes which result from use, on the one hand, of Eq. (47) of Part I for definition of the relationship which holds between the geometrical features of the sought contour and the local pressure coefficient, $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$, and, on the other hand, from use of the formula which is supposed to hold strictly merely for the case of two-dimensional flow (Ackeret's airfoil theory).

Although this result does constitute a notable simplification to the practical solution of the originally set problem, it may appear to be disappointing and of little consequence, especially after going through all the details required in the three-dimensional treatment. Nevertheless, it should not be interpreted as being of only minor importance, because one will also be able to observe, on the basis of these same numerical results, to be presented below, that the local pressures exhibit very sizeable differences between the case when formula (47) of Part I is employed to relate the geometrical properties of the duct's contour to the pressure coefficients, in contrast to the case in which the Ackeret formula is used.

It is worth pointing out in this regard that the numerical values of βE and of ℓ_1, selected for purposes of illustration herein, are such as

to make it probable that the above-mentioned overall conclusion will still hold true for all other cases which may arise in actual practice, inasmuch as any cases of practical interest which are likely to be met with will, in general, be found to be characterized by values of βB and of $\mathcal{L}_1$ that will be smaller in magnitude than the ones premised in Eq. (1).

2. Evaluation of the Γ^* Function, Appearing in Eq. (38') of Part I

The first step in carrying out the computations for the determination of the optimal duct contour is the evaluation of the resolvent kernel Γ^* of the governing integral equation, given as Eq. (37) of Part I. The expression for the resolvent kernel to be used is

$$\Gamma^*(x) = H(x) - \frac{1}{\sqrt{1+\beta B}} H^{(2)}(x) + \frac{1}{1+\beta B} H^{(3)}(x) - \frac{1}{(1+\beta B)^{3/2}} H^{(4)}(x) + \frac{1}{(1+\beta B)^2} H^{(5)}(x) \ . \tag{2}$$

where $H(x) = \frac{dK}{dx}$, and where the expression for $K(x)$ is given as Eq. (34) of Part I. In the specific case under study, one finds, by insertion of the numerical value of βB selected in Eq. (1), that

$$H(x) = 0.45016 \ \frac{(1+x)\,\mathbb{E} - \mathbb{F}}{x\sqrt{x+1.6}} \tag{3}$$

and the moduli of the complete elliptic integrals $\mathbb{E}$ and $\mathbb{F}$ appearing herein are reduced to just $\sqrt{\frac{x}{1.6+x}}$ in both cases.

The values for $H(x)$ and for the successively higher orders of the iterative kernels $H^{(2)}$, $H^{(3)}$, $H^{(4)}$, and $H^{(5)}$, as well as the pertinent values for the resolvent kernel Γ^* itself, have been determined for a goodly number of points within the range of x-values running from 0 to 4.2, and the results have been entered in the appended Table I.

TABLE I

RESOLVENT KERNEL FOR EQ.(37) OF PART I, TOGETHER WITH ITS INTERVENING ITERATIVE KERNELS UNDER ASSUMPTION THAT $\beta B = 0.25$

x	H(x)	$H^{(2)}(x)$	$H^{(3)}(x)$	$H^{(4)}(x)$	$H^{(5)}(x)$	$\Gamma^*(x)$
0	0.38433	0	0	0	0	0.38433
0.1	0.37109	0.01409	0.00026	0	0	0.35870
0.2	0.35913	0.02692	0.00100	0.00003	0	0.33583
0.3	0.34822	0.03865	0.00214	0.00008	0	0.31530
0.4	0.33824	0.04941	0.00360	0.00018	0.00001	0.29681
0.5	0.32903	0.05932	0.00534	0.00033	0.00002	0.28001
0.6	0.32056	0.06846	0.00732	0.00053	0.00003	0.26481
0.7	0.31266	0.07692	0.00948	0.00079	0.00005	0.25090
0.8	0.30535	0.08476	0.01180	0.00110	0.00008	0.23824
0.9	0.29850	0.09205	0.01425	0.00148	0.00012	0.22659
1	0.29209	0.09884	0.01681	0.00192	0.00017	0.21588
1.2	0.28042	0.11563	0.02311	0.00309	0.00032	0.19348
1.4	0.27003	0.13035	0.02975	0.00453	0.00053	0.17434
1.6	0.26071	0.14333	0.03660	0.00621	0.00080	0.15786
1.8	0.25228	0.15487	0.04356	0.00813	0.00115	0.14353
2	0.24462	0.16517	0.05058	0.01027	0.00158	0.13101
2.2	0.23763	0.17442	0.05759	0.01259	0.00208	0.12001
2.4	0.23118	0.18276	0.06455	0.01509	0.00267	0.11026
2.6	0.22524	0.19031	0.07143	0.01775	0.00333	0.10159
2.8	0.21974	0.19717	0.07822	0.02054	0.00408	0.09388
3	0.21462	0.20342	0.08489	0.02346	0.00490	0.08694
3.3	0.20755	0.21664	0.09805	0.02919	0.00655	0.07552
3.6	0.20116	0.22825	0.11075	0.03518	0.00840	0.06582
3.9	0.19532	0.23850	0.12299	0.04137	0.01045	0.05748
4.2	0.18995	0.24762	0.13478	0.04772	0.01268	0.05026

$$\Gamma^*(x) = H(x) - 0.89443\, H^{(2)}(x) + 0.8\, H^{(3)}(x) - 0.71554\, H^{(4)}(x) + 0.64\, H^{(5)}(x)$$

to make it probable that the above-mentioned overall conclusion will still hold true for all other cases which may arise in actual practice, inasmuch as any cases of practical interest which are likely to be met with will, in general, be found to be characterized by values of βB and of ℓ_1 that will be smaller in magnitude than the ones premised in Eq. (1).

2. Evaluation of the Γ^* Function, Appearing in Eq. (38') of Part I

The first step in carrying out the computations for the determination of the optimal duct contour is the evaluation of the resolvent kernel Γ^* of the governing integral equation, given as Eq. (37) of Part I. The expression for the resolvent kernel to be used is

$$\Gamma^*(x) = H(x) - \frac{1}{\sqrt{1+\beta B}} H^{(2)}(x) + \frac{1}{1+\beta B} H^{(3)}(x) - \frac{1}{(1+\beta B)^{3/2}} H^{(4)}(x) + \frac{1}{(1+\beta B)^2} H^{(5)}(x) \ . \tag{2}$$

where $H(x) = \frac{dK}{dx}$, and where the expression for $K(x)$ is given as Eq. (34) of Part I. In the specific case under study, one finds, by insertion of the numerical value of βB selected in Eq. (1), that

$$H(x) = 0.45016 \ \frac{(1+x)\,\mathbb{E} - \mathbb{F}}{x\sqrt{x+1.6}} \tag{3}$$

and the moduli of the complete elliptic integrals $\mathbb{E}$ and $\mathbb{F}$ appearing herein are reduced to just $\sqrt{\frac{x}{1.6+x}}$ in both cases.

The values for $H(x)$ and for the successively higher orders of the iterative kernels $H^{(2)}$, $H^{(3)}$, $H^{(4)}$, and $H^{(5)}$, as well as the pertinent values for the resolvent kernel Γ^* itself, have been determined for a goodly number of points within the range of x-values running from 0 to 4.2, and the results have been entered in the appended Table I.

3. Determination of the Supersonic Source Distribution Which Will Produce the Required Radial Components of Velocity Along the Duct Contour

The just determined values for the resolvent kernel $\Gamma^*(x)$ may now be used to find the distribution of supersonic sources which are located on the duct's axis and the strength of which is to vary in the proper way in order that the radial velocity component of the induced velocity along the sought duct's surface connecting the end-points N_1 and N_2 shall be equal to the values defined by the relations

$$U_\infty \eta_n(z) = \frac{\cos n\theta}{1+\beta B \frac{\ell_1}{\pi}\theta} U_\infty$$

at the various points of the duct contour running from N_1 to N_2, and approximated by a seven term series, for which $n = 0,1,2,3,4,5,6$

$$U_\infty \eta_{m+k}(z) = \frac{\sin k\theta}{1+\beta B \frac{\ell_1}{\pi}\theta} U_\infty$$

where this time the radial components throughout the N_1 to N_2 interval are approximated by a six term series, for which $k = 1,2,3,4,5,6$

The way this source distribution must behave is found by first evaluating the $\overset{\circ}{v}(s)$ function defined as

$$v(s) = \frac{g^*(0)}{\sqrt{s}} + \int_0^s \frac{\dot{g}^*(z)\,dz}{\sqrt{s-z}}$$

where it is now true that

$$g^*(z) = \frac{1}{B}(1+\beta B z)\eta_n = \frac{1}{B}\begin{cases}\cos n\theta \\ \sin k\theta\end{cases}.$$

Then it follows, upon using the respective cosine and sine formulations, that one must have

$$g^*(0) = \frac{1}{B}\begin{cases}1 \\ 0\end{cases}$$

and

$$\frac{\dot{g}^*(z)\,dz}{\sqrt{s-z}} = \frac{1}{B}\sqrt{\frac{\pi}{\ell_1}}\begin{cases}\dfrac{-n\sin n\theta}{\sqrt{\theta'-\theta}}\,d\theta \\ k\dfrac{\cos k\theta}{\sqrt{\theta'-\theta}}\,d\theta\end{cases}$$

(where $\frac{\ell_1}{\pi}\theta'$ has been inserted for s)

and thus it may be deduced that

$$B\dot{v}(s) = \begin{cases} \frac{1}{\sqrt{s}} - \sqrt{\frac{\pi}{\ell_1}} \cdot n\int_0^{\theta'} \frac{\sin n\theta}{\sqrt{\theta'-\theta}}\, d\theta \\ \sqrt{\frac{\pi}{\ell_1}}\, \kappa\int_0^{\theta'} \frac{\cos \kappa\theta}{\sqrt{\theta'-\theta}}\, d\theta \end{cases}$$

Now, as is well known,

$$\int_0^{\theta'} \frac{\sin n\theta}{\sqrt{\theta'-\theta}}\, d\theta = \sqrt{\frac{2\pi}{n}}\left[\sin(n\theta')\cdot \mathcal{C}(n\theta') - \cos(n\theta')\mathcal{S}(n\theta')\right]$$

and

$$\int_0^{\theta'} \frac{\cos \kappa\theta}{\sqrt{\theta'-\theta}}\, d\theta = \sqrt{\frac{2\pi}{\kappa}}\left[\cos(\kappa\theta')\mathcal{C}(\kappa\theta') + \sin(\kappa\theta')\mathcal{S}(\kappa\theta')\right]$$

where the symbols $\mathcal{C}(n\theta')$ and $\mathcal{S}(n\theta')$ denote, respectively, the Fresnel integrals of the first and second kinds, with the argument which is represented by the quantity within the parentheses.

By use of these Fresnel integrals, therefore, the expressions for $B\,\dot{v}(s)$ may be evaluated; the results are listed in Tables II and III. Once these values have been obtained, one may easily find the corresponding values for $B\,w(s)$, because one has simply that

$$B\,w(s) = \frac{\sqrt{2}}{\pi}\,\frac{B}{\sqrt{1-\beta B}}\cdot\frac{\dot{v}(s)}{\sqrt{1+\beta B s}} = 0.5198\,\frac{B\dot{v}(s)}{\sqrt{1+0.25 s}} .$$

This calculation has been carried out, and the results are given in Tables IV and V.

TABLE II

EVALUATION FOR THE $B\dot{v}(s)$ FUNCTION APPEARING AS INTERMEDIARY STAGE BETWEEN CONVERSION OF DUCT'S LOCAL SLOPE TO CORRESPONDING SOURCE INTENSITY FUNCTION, FOR THE CASE OF THE COSINE FORMULATION DESCRIBING THE RADIAL COMPONENTS OF VELOCITY; i.e., for $\eta_n = \dfrac{\cos n\theta}{1+\beta B\frac{\ell_i}{\pi}\theta}$

s \ n	0	1	2	3	4	5	6
0	∞	∞	∞	∞	∞	∞	∞
0.1	3.16228	3.11873	2.98793	2.77098	2.47211	2.09548	1.64638
0.2	2.23607	2.11279	1.74805	1.16417	0.39621	-0.49148	-1.43143
0.3	1.82574	1.59982	0.95054	-0.03474	-1.16876	-2.42158	-3.48744
0.4	1.58114	1.23606	0.28017	-1.01217	-2.40986	-3.51296	-4.01278
0.5	1.41421	0.93712	-0.31084	-1.87575	-3.14209	-3.56185	-2.83643
0.6	1.29099	0.67213	-0.82643	-2.46599	-3.27643	-2.58930	-0.52113
0.7	1.19523	0.42788	-1.32912	-2.78369	-2.77186	-0.87303	2.03673
0.8	1.11803	0.19810	-1.70403	-2.83747	-1.76027	1.11759	3.86237
0.9	1.05409	-0.02006	-2.01348	-2.53900	-0.42550	2.84295	4.28371
1	1	-0.21980	-2.22178	-2.00566	-0.99961	3.83666	3.08986
1.1	0.9535				2.2664	3.8296	0.7359
1.2	0.91287	-0.58438	-2.31677	-0.36849	3.15361	2.82064	-1.86681
1.3	0.8771				3.5265	1.0781	-3.7087
1.4	0.84515	-0.93984	-1.96007	1.44018	3.30699	-0.93155	-4.12099
1.5	0.8165				2.5228	-2.6730	-2.9455
1.6	0.79057	-1.20493	-1.24469	2.73111	1.31045	-3.68082	-0.61331
1.7	0.7670				-0.1207	-3.6854	1.9867
1.8	0.74536	-1.42374	-0.30086	3.02905	-1.52424	-2.68821	3.83854
1.9	0.7255				-2.6455	-0.9566	4.2361
2	0.70711	-1.57104	0.70683	2.18486	-3.29222	1.04308	3.04435
2.1	0.6901				-3.3648	2.7742	0.7171
2.2	0.67420	-1.63106	1.50254	0.52036	-2.85008	3.77749	-1.37490
2.3	0.6594				-1.8371	3.7798	-3.7428
2.4	0.64550	-1.63821	2.22994	-1.32003	-0.50125	2.77910	-4.14910
2.5	0.6325				0.9328	1.0444	-2.9513
2.6	0.62017	-1.53518	2.49345	-2.62286	2.22415	-0.95941	-0.61799
2.7	0.6086				3.1340	-2.6947	1.9598
2.8	0.59761	-1.38599	2.33838	-2.91398	3.50672	-3.69676	3.81919
2.9	0.5872				3.2777	-3.6965	4.2352
3	0.57735	-1.15797	1.78393	-2.08228	2.48570	-2.69436	3.04143

TABLE III

EVALUATION FOR THE $B\dot{v}(s)$ FUNCTION APPEARING AS INTERMEDIARY STAGE BETWEEN CONVERSION OF DUCT'S LOCAL SLOPE TO CORRESPONDING SOURCE INTENSITY FUNCTION, FOR THE CASE OF THE SINE FORMULATION DESCRIBING THE RADIAL COMPONENTS OF VELOCITY; i.e., FOR $\eta_{m+k} = \dfrac{\sin k\theta}{1+\beta B \frac{\ell_1 \theta}{\pi}}$

s \ k	1	2	3	4	5	6
0	0	0	0	0	0	0
0.1	0.6283	1.2474	1.8451	2.4103	2.9325	3.4032
0.2	0.8821	1.7040	2.4064	2.9360	3.2409	3.2976
0.3	1.0653	1.9648	2.5498	2.6925	2.3903	1.6104
0.4	1.2049	2.0760	2.3317	1.8817	0.7416	-0.8904
0.5	1.3117	2.0497	1.8515	0.6633	-1.2066	-3.1806
0.6	1.3894	1.9039	1.1387	-0.7270	-2.9036	-4.3332
0.7	4.4408	1.6728	0.2758	-2.0414	-3.8779	-3.8940
0.8	1.4680	1.3306	-0.6296	-3.0384	-3.8565	-2.0226
0.9	1.4721	0.9298	-1.5044	-3.5381	-2.8372	0.5784
1	1.4494	0.4690	-2.2490	-3.4494	-1.0878	2.9280
1.1				-2.7831	0.9278	4.1292
1.2	1.3462	-0.5141	-3.0640	-1.6515	2.6729	3.7242
1.3				-0.2513	3.6829	1.8744
1.4	1.1829	-1.4435	-2.7534	1.1782	3.6889	-0.7086
1.5				2.3907	2.6915	-3.0390
1.6	0.9409	-2.1484	-1.4302	3.1790	0.9602	-4.2282
1.7				3.4087	-1.0412	-3.8160
1.8	0.6574	-2.5018	0.4090	3.0408	-2.7742	-1.9572
1.9				2.1394	-3.7743	0.6366
2	0.3317	-2.4391	2.0704	0.8588	-3.7721	2.9718
2.1				-0.5786	-2.7676	4.1646
2.2	-0.0143	-1.9680	2.9198	-1.9219	-1.0303	3.7578
2.3				-2.9389	0.9760	1.9044
2.4	-0.3635	-1.1677	2.6334	-3.4523	2.7129	-0.6852
2.5				-3.3735	3.7168	-3.0198
2.6	-0.7082	-0.1774	1.3254	-2.7180	3.7185	-4.2078
2.7				-1.5981	2.7167	-3.7968
2.8	-1.0207	0.8331	-0.5011	-0.2060	0.9815	-1.9440
2.9				1.2174	-1.0220	0.6474
3	-1.2985	1.6905	-2.1489	2.4265	-2.7567	2.9862

TABLE IV

EVALUATION OF THE PART OF THE INTEGRAL EQUATION, EQ. (36) OF PART I, WHICH MAKES IT NONHOMOGENEOUS, IN THE CASE OF THE COSINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY; i.e.,

TABULATED VALUES OF Bw(s), IN CASE WHERE $\eta_n = \dfrac{\cos n\theta}{1+\beta B \frac{\ell_l}{\pi}\theta}$

s \ n	0	1	2	3	4	5	6
0	∞	∞	∞	∞	∞	∞	∞
0.1	1.62351	1.6011	1.5340	1.4226	1.2692	1.0758	0.8453
0.2	1.13436	1.0718	0.8868	0.5906	0.2010	-0.2493	-0.7261
0.3	0.91524	0.8020	0.4765	-0.0174	-0.5859	-1.2139	-1.7483
0.4	0.78361	0.6126	0.1389	-0.5016	1.1943	-1.7410	-1.9887
0.5	0.69310	0.4593	-0.1523	-0.9193	-1.5399	-1.7457	-1.3901
0.6	0.62574	0.3258	-0.4006	-1.1953	-1.5881	-1.2550	-0.2526
0.7	0.57323	0.2052	-0.6374	-1.3351	-1.3295	-0.4187	0.9768
0.8	0.53051	0.0940	-0.8085	-1.3464	-0.8352	0.5303	1.8327
0.9	0.49500	-0.0094	-0.9455	-1.1923	-0.1998	1.3350	2.0116
1	0.46490	-0.1022	-1.0329	-0.9324	0.4647	1.7837	1.4365
1.1	0.43890				1.0432	1.7628	0.3387
1.2	0.41618	-0.2664	-1.0562	-0.1680	1.4377	1.2859	-0.8511
1.3	0.39610				1.5926	0.4869	-1.6748
1.4	0.37804	-0.4204	-0.8767	0.6442	1.4792	-0.4167	-1.8433
1.5	0.36195				1.1184	-1.1849	-1.3057
1.6	0.34730	-0.5293	-0.5468	1.1998	0.5757	-1.6170	-0.2697
1.7	0.33395				-0.0526	-1.6046	0.8648
1.8	0.32170	-0.6145	-0.1299	1.3073	-0.6579	-1.1602	1.6567
1.9	0.31051				-1.1323	-0.4094	1.8131
2	0.30010	-0.6667	0.3000	0.9273	-1.3972	0.4427	1.2920
2.1	0.29046				-1.4162	1.1677	0.3018
2.2	0.28148	-0.6810	0.6691	0.2173	-1.1899	1.5771	-0.7828
2.3	0.27312				-0.7609	1.5656	-1.5503
2.4	0.26530	-0.6733	0.9165	-0.5425	-0.2060	1.1422	-1.7053
2.5	0.25793				0.3804	0.4259	-1.2035
2.6	0.25098	-0.6215	1.0091	-1.0615	0.9001	-0.3883	-0.2501
2.7	0.24441				1.2586	-1.0822	0.7871
2.8	0.23821	-0.5525	0.9321	-1.1615	1.3978	-1.4735	1.5223
2.9	0.23241				1.2973	-1.4631	1.6763
3	0.22684	-0.4550	0.7009	-0.8181	-0.9766	-1.0586	1.1950

TABLE V

EVALUATION OF THE PART OF THE INTEGRAL EQUATION, EQ.(36) OF PART I, WHICH MAKES IT NONHOMOGENEOUS, IN THE CASE OF THE SINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY; i.e.,

TABULATED VALUES OF B w(s), IN CASE WHERE $\eta_{m+k} = \dfrac{\sin k\theta}{1+\beta B\frac{\ell_1}{\pi}\theta}$

s \ k	1	2	3	4	5	6
0	0	0	0	0	0	0
0.1	0.3226	0.6404	0.9473	1.2374	1.5055	1.7472
0.2	0.4475	0.8644	1.2207	1.4893	1.6440	1.6728
0.3	0.5341	0.9850	1.2783	1.3499	1.1984	0.8074
0.4	0.5972	1.0289	1.1556	0.9326	0.3676	-0.4413
0.5	0.6428	1.0045	0.9073	0.3251	-0.5913	-1.5587
0.6	0.6735	0.9228	0.5519	-0.3524	-1.4074	-2.1003
0.7	0.6910	0.8022	0.1323	-0.9790	-1.8597	-1.8674
0.8	0.6966	0.6314	-0.2988	-1.4418	-1.8300	-0.9598
0.9	0.6913	0.4367	-0.7065	-1.6616	-1.3324	0.2716
1	0.6738	0.2180	-1.0456	-1.6036	-0.5057	1.3612
1.1				-1.2811	0.4271	1.9008
1.2	0.6137	-0.2344	-1.3968	-0.7529	1.2185	1.6977
1.3				-0.1135	1.6630	0.8464
1.4	0.5291	-0.6457	-1.2317	0.5270	1.6502	-0.3170
1.5				1.0597	1.1931	-1.3471
1.6	0.4134	-0.9439	-0.6283	1.3966	0.4218	-1.8576
1.7				1.4843	-0.4534	-1.6616
1.8	0.2838	-1.0799	0.1765	1.3125	-1.1977	-0.8448
1.9				0.9157	-1.6154	0.2725
2	0.1408	-1.0352	0.8787	0.3645	-1.6009	1.2613
2.1				-0.2436	-1.1650	1.7530
2.2	-0.0060	-0.8216	1.2190	-0.8024	-0.4302	1.5689
2.3				1.2172	0.4042	0.7887
2.4	-0.1494	-0.4799	1.0822	-1.4187	1.1149	-0.2816
2.5				-1.3756	1.5156	-1.2314
2.6	-0.2866	-0.0718	0.5363	-1.0999	1.5047	-1.7027
2.7				-0.6419	1.0911	-1.5249
2.8	-0.4069	0.3321	-0.1998	-0.0821	0.3913	-0.7749
2.9				0.4818	-0.4045	-0.2562
3	-0.5102	0.6642	-0.8443	0.9534	-1.0831	1.1733

The next step is to insert the above-derived values for w(s) in Eq. (37') of Part I in order to find the values of $B\,h^*(s)$. The latter function defines the way in which the pertinent supersonic source distribution has to be apportioned along the duct's axis. The values for this source intensity function for the specific case under study have been determined and the results are tabulated in Tables VI and VII; in addition, they are plotted in Figs. 1 and 2, as thirteen individual curves, which correspond to the thirteen harmonic components used to approximate the radial velocity distribution.

4. <u>Computation of the Axial Components of the Induced Velocity Arising from the Individual Supersonic Source Distributions</u>

The axial components of the velocity field induced at the points on the meridional contour running between N_1 and N_2 are now to be calculated in the case of each one of the thirteen supersonic source distributions determined above. The equation which defines these axial components of the induced flow in terms of the source intensity distribution is

$$-\frac{B}{U_\infty}\,\frac{\partial \Phi_n}{\partial x} = B\int_0^z \frac{h^*(s)\,ds}{\sqrt{(z-s)\left(\frac{1+\beta B}{1-\beta B}\,z - s + \frac{2}{1-\beta B}\right)}} \quad . \tag{4}$$

This formula, however, is not convenient to use in performing the actual numerical evaluations because of the singularity possessed by the integrand at the point for which z = s = 0 when one has to deal with the cosine formulation for the radial components, given as $\eta_n = \dfrac{\cos n\theta}{1+\beta B\,\frac{l_1}{\pi}\,\theta}$.

Nevertheless this difficulty may be easily side-stepped by converting the expression (4) into the following rewritten version, which readily allows one to carry out the desired calculation even in this troublesome case of the

TABLE VI

ADJUSTED SUPERSONIC SOURCE INTENSITY FUNCTION, $Bh^*(s)$, IN THE CASE OF THE COSINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY, $\eta_n = \dfrac{\cos n\theta}{1+\beta B \frac{\ell}{\pi}\theta}$

s \ n	0	1	2	3	4	5	6
0	∞	∞	∞	∞	∞	∞	∞
0.1	1.52062	1.49821	1.43111	1.31971	1.16631	0.97291	0.74241
0.2	0.99547	0.93428	0.75335	0.46372	0.08287	-0.35706	-0.82248
0.3	0.75462	0.64541	0.33168	-0.14376	-0.68915	-1.29070	-1.79879
0.4	0.60846	0.44555	-0.00500	-0.61103	-1.26304	-1.76733	-1.95627
0.5	0.50796	0.28774	-0.28611	-1.00004	-1.56327	-1.72854	-1.33109
0.6	0.43366	0.15418	-0.51750	-1.23926	-1.56460	-1.18645	-0.17437
0.7	0.37641	0.03704	-0.73203	-1.34101	-1.26636	-0.33140	1.03883
0.8	0.33059	-0.06784	-0.87716	-1.31486	-0.74643	0.60976	1.85095
0.9	0.29323	-0.16260	-0.98621	-1.12846	-0.10303	1.38339	1.97589
1	0.26223	-0.24490	-1.04499	-0.84450	0.55141	1.78694	1.35690
1.1	0.23639	-0.31453	-1.03098	-0.45616	1.10472	1.71946	0.24131
1.2	0.21468	-0.38416	-1.01698	-0.06781	1.46483	1.20742	-0.93562
1.3	0.19590	-0.44593	-0.90520	0.32681	1.57912	0.39041	-1.72115
1.4	0.17971	-0.50770	-0.79343	0.72143	1.42888	-0.50765	-1.84275
1.5	0.16563	-0.54616	-0.61618	0.97347	1.03975	-1.25061	-1.26686
1.6	0.15334	-0.58462	-0.43894	1.22551	0.48170	-1.64514	-0.21461
1.7	0.14244	-0.61118	-0.22836	1.24971	-0.14727	-1.59338	0.90907
1.8	0.13287	-0.63775	-0.01779	1.27391	-0.73895	-1.11792	1.66871
1.9	0.12432	-0.64844	0.18968	1.06135	-1.18868	-0.35212	1.78444
2	0.11672	-0.65912	0.39714	0.84879	-1.42241	0.49538	1.22991
2.1	0.10985	-0.65214	0.56664	0.48574	-1.40946	1.19981	0.22543
2.2	0.10376	-0.64517	-0.73615	0.12268	-1.15585	1.57730	-0.84934
2.3	0.09799	-0.62890	0.84034	-0.24918	-0.70822	1.53192	-1.58862
2.4	0.09313	-0.61264	0.94452	-0.62104	-0.14606	1.08197	-1.70691
2.5	0.08853	-0.57633	0.97027	-0.86086	0.43517	0.35266	-1.17471
2.6	0.08436	-0.54002	0.99602	-1.10067	0.93889	-0.45797	-0.20785
2.7	0.08047	-0.49776	0.93918	-1.12754	1.27367	-1.13335	0.82176
2.8	0.07700	-0.45549	0.88234	-1.15441	1.38586	-1.49650	1.53222
2.9	0.07381	-0.40133	0.75340	-0.96468	1.25973	-1.45611	1.65422
3	0.07086	-0.34717	0.62447	-0.77496	0.91915	-1.02753	1.14612

TABLE VII

ADJUSTED SUPERSONIC SOURCE INTENSITY FUNCTION, $Bh^*(s)$, IN THE CASE OF THE SINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY,

$$\eta_{m+k} = \frac{\sin k\theta}{1+\beta B\frac{\ell_1}{\pi}\theta}$$

s \ k	1	2	3	4	5	6
0	0	0	0	0	0	0
0.1	0.31719	0.69267	0.93145	1.21665	1.48025	1.71791
0.2	0.43018	0.83041	1.17137	1.42659	1.57045	1.59129
0.3	0.50274	0.92467	1.19368	1.24766	1.08620	0.69364
0.4	0.55083	0.94221	1.04002	0.80308	0.23990	-0.55202
0.5	0.58117	0.89356	0.76911	0.18606	-0.70648	-1.63160
0.6	0.59685	0.79129	0.40189	-0.48145	-1.48548	-2.11491
0.7	0.59991	0.65475	-0.01751	-1.03047	-1.88605	-1.82477
0.8	0.59196	0.47341	-0.43661	-1.50295	-1.80338	-0.88105
0.9	0.57427	0.27399	-0.82193	-1.67663	-1.26470	0.35403
1	0.54555	0.05645	-1.13056	-1.57426	-0.41817	1.41593
1.1	0.50754	-0.15748	-1.26650	-1.21610	0.50968	1.90886
1.2	0.46952	-0.37140	-1.40244	-0.66603	1.27443	1.65886
1.3	0.42130	-0.55862	-1.28785	-0.02111	1.67904	0.77812
1.4	0.37308	-0.74584	-1.17325	0.60870	1.62433	-0.38715
1.5	0.31249	-0.87086	-0.85320	1.11743	1.13457	-1.39203
1.6	0.25190	-0.99588	-0.53314	1.42192	0.34793	-1.86091
1.7	0.18757	-1.03814	-0.13080	1.47468	-0.52206	-1.62298
1.8	0.12324	-1.08041	0.27154	1.27151	-1.24292	-0.76512
1.9	0.05518	-1.03472	0.60622	0.85203	-1.62583	0.34052
2	-0.01288	-0.98903	0.94089	0.29029	-1.57469	1.30739
2.1	-0.08018	-0.86465	1.08583	-0.31489	-1.10986	1.76350
2.2	-0.14747	-0.74027	1.23076	-0.85828	-0.36132	1.54249
2.3	-0.21084	-0.55980	1.13836	-1.24875	0.46962	0.73831
2.4	-0.27420	-0.37932	1.04595	-1.42108	1.16055	-0.33466
2.5	-0.33264	-0.17433	0.76438	-1.34919	1.53135	-1.26585
2.6	-0.39107	0.03066	0.48281	-1.04986	1.48863	-1.70523
2.7	-0.43972	0.22556	0.10910	-0.57711	1.04967	-1.49459
2.8	-0.48837	0.42046	-0.26461	-0.01362	0.33731	-0.72308
2.9	-0.52770	0.57330	-0.57409	0.54344	-0.45553	0.31071
3	-0.56703	0.72615	-0.88356	0.99751	-1.11707	1.21142

HARMONIC COMPONENTS OF THE ADJUSTED SUPERSONIC SOURCE INTENSITY FUNCTION, $Bh^*(s)$, IN THE CASE OF THE COSINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY, $\eta_n = \dfrac{\cos n\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}$

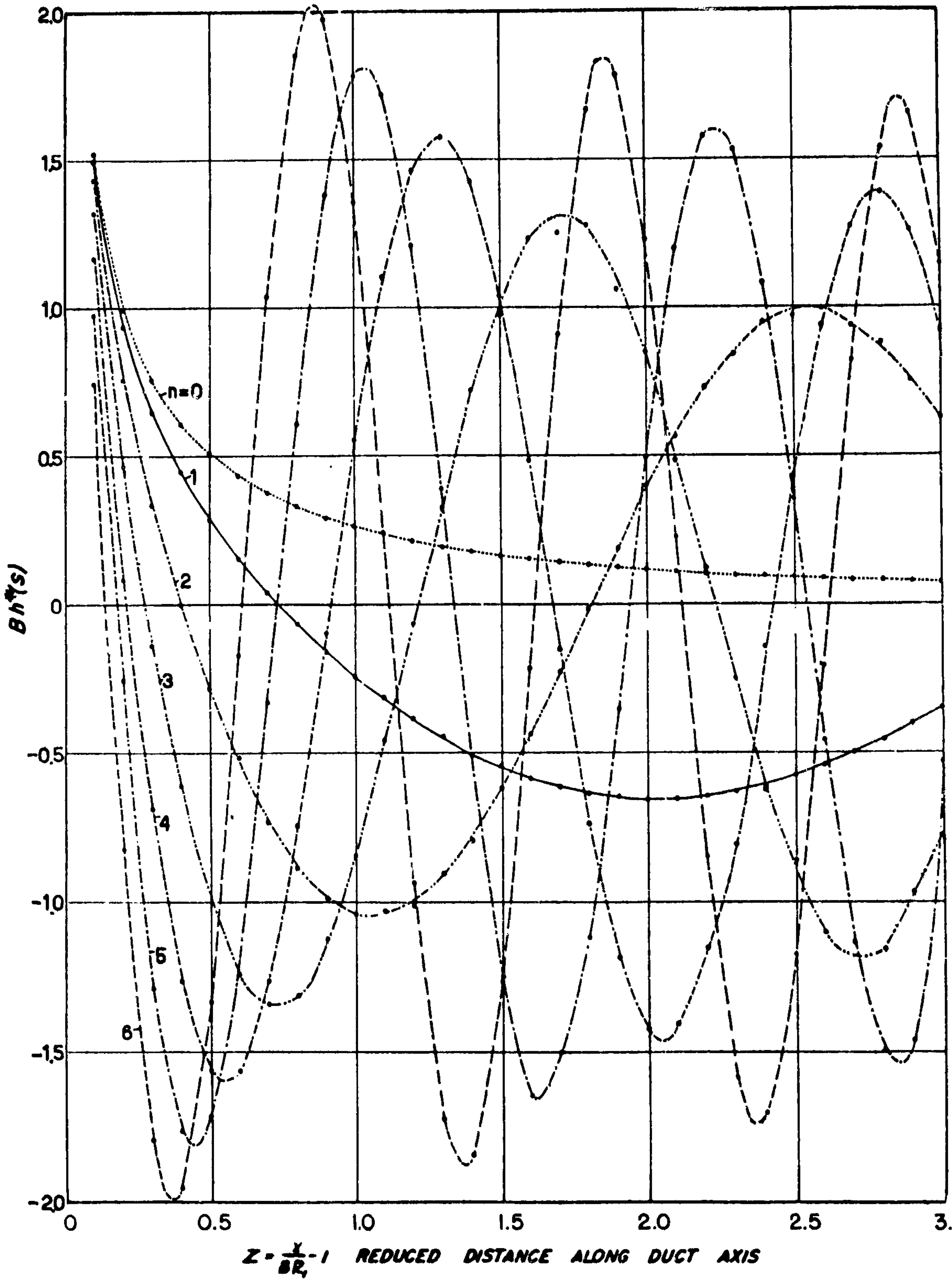

FIGURE 1

HARMONIC COMPONENTS OF THE ADJUSTED SUPERSONIC SOURCE INTENSITY FUNCTION, $Bh^*(s)$, IN THE CASE OF THE SINE FORMULATION FOR THE RADIAL COMPONENTS OF VELOCITY, $\eta_{m+k} = \dfrac{\sin k\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}$

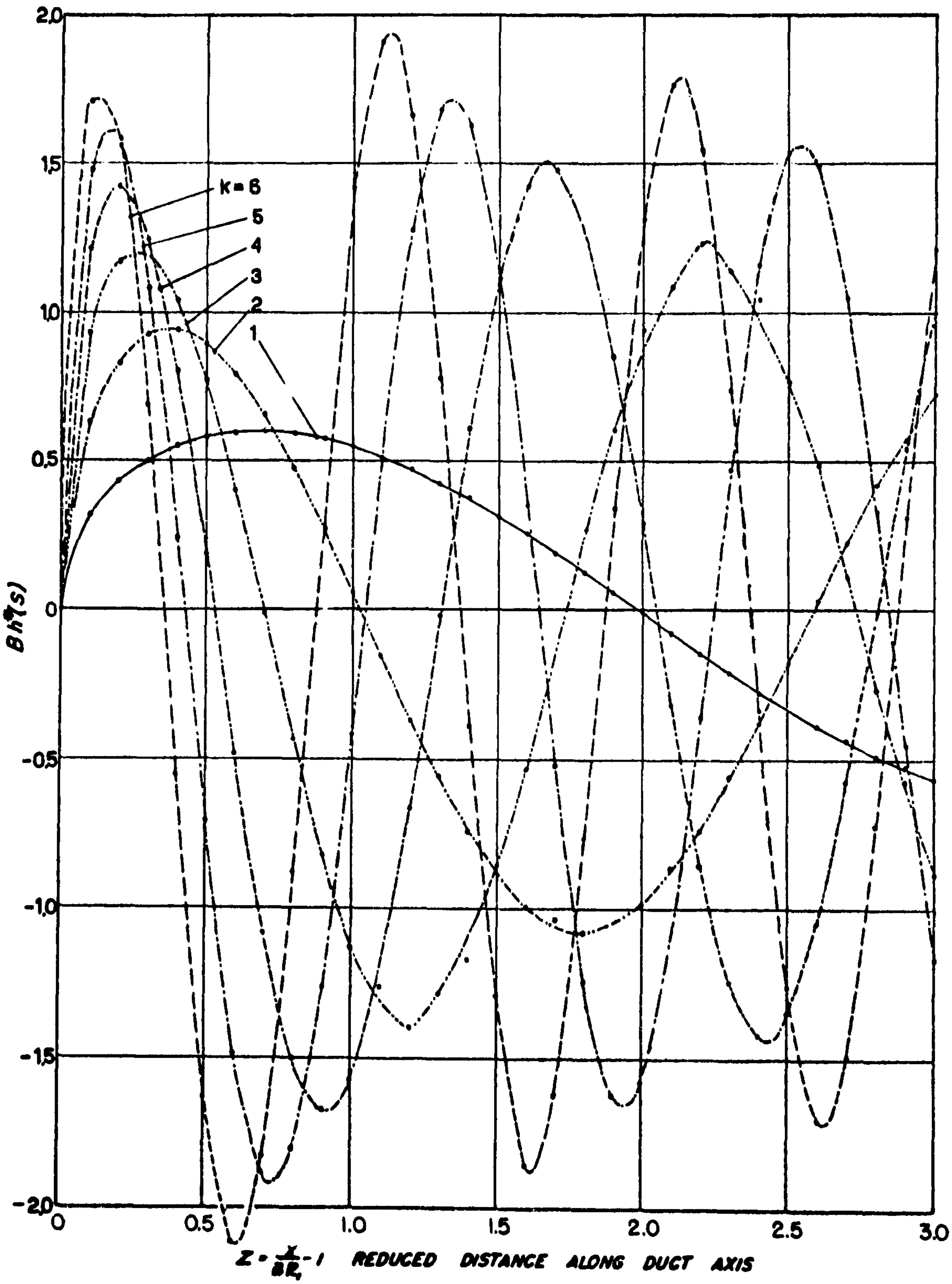

FIGURE 2

cosine formulation:

$$-\frac{B}{U_\infty}\frac{\partial \Phi_n}{\partial x} = B\int_0^z \frac{h^{**}(s)\,ds}{\sqrt{z-s}} + \frac{\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{1-\beta B}}\int_0^z \frac{ds}{\sqrt{s}\sqrt{z-s}\sqrt{\frac{1+\beta B}{1-\beta B}\cdot z - s + \frac{2}{1-\beta B}}} \tag{5}$$

where the $h^{**}(s)$ function is defined as

$$Bh^{**}(s) = \frac{Bh^{*}s - \frac{\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{1-\beta B}}\cdot\frac{1}{\sqrt{s}}}{\sqrt{\frac{1+\beta B}{1-\beta B}\,z - s + \frac{2}{1-\beta B}}} \quad . \tag{6}$$

This newly introduced $h^{**}(s)$ function is now continuous at the point for which $z = s = 0$, and thus the indicated quadrature involving this function that appears on the right hand side of Eq. (5) may be carried out numerically without further difficulty. The second integral appearing on the right hand side of Eq. (5) may be evaluated, of course, in closed form, by use of the complete elliptic integral of the first kind. Thus the evaluation of the axial velocity components in the case where the cosine formulation is being treated may be carried out by use of the relations:

$$-\frac{B}{U_\infty}\;\frac{\partial \Phi_n}{\partial x} = -F_n(z) \tag{7}$$

where

$$F_n(z) = \int_0^z \frac{Bh^{**}(s)}{\sqrt{z-s}}\,ds + \frac{1.0396}{\sqrt{1.6667z + 2.6667}}\,F\left(\sqrt{\frac{z}{1.6667z+2.6667}}\right) \tag{8}$$

in which B $H^{**}(s)$ has the specific form

$$Bh^{**}(s) = \frac{Bh^{*}(s) - 0.5198\cdot\frac{1}{\sqrt{s}}}{\sqrt{1.6667z - s + 2.6667}} \quad .$$

These evaluations for $F_n(z)$ have been performed in the case of interest and the results are presented in Table VIII, and, in addition, the results

TABLE VIII

AXIAL COMPONENTS OF INDUCED VELOCITIES AT THE DUCT'S SURFACE, $F_n = \frac{+B}{U_\infty}\frac{\partial \phi_n}{\partial x}$, CALCULATED BY MEANS OF Eq.(47) OF PART I, AND COMPARISON OF THESE VALUES WITH THOSE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_n^* = \frac{\cos n\theta}{1+\beta B \frac{1}{\pi}\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE COSINE FORMULATION FOR WHICH $\eta_x = \frac{\cos n\theta}{1+\beta B \frac{1}{\pi}\theta}$

z \ n	0	1	2	3	4	5	6
0	$F_0 = 1$ $F_0^* = 1$	$F_1 = 1$ $F_1^* = 1$	$F_2 = 1$ $F_2^* = 1$	$F_3 = 1$ $F_3^* = 1$	$F_4 = 1$ $F_4^* = 1$	$F_5 = 1$ $F_5^* = 1$	$F_6 = 1$ $F_6^* = 1$
0.1	0.94356 0.9756	0.93713 0.9703	0.91788 0.9543	0.88592 0.9279	0.84191 0.8913	0.78642 0.8449	0.72029 0.7893
0.2	0.88898 0.9524	0.86747 0.9316	0.80372 0.8700	0.70094 0.7705	0.56449 0.6372	0.40387 0.4762	0.22914 0.2943
0.3	0.83982 0.9302	0.79513 0.8847	0.66546 0.7526	0.46485 0.5468	0.22256 0.2875	-0.05144 0	-0.30866 -0.2875
0.4	0.79525 0.9091	0.72031 0.8305	0.50889 0.6083	0.20682 0.2809	-0.13645 -0.0950	-0.46055 -0.4545	-0.68734 -0.7355
0.5	0.75467 0.8889	0.64338 0.7698	0.34215 0.4444	-0.06030 0	-0.45125 -0.4444	-0.72034 -0.7698	-0.77822 -0.8888
0.6	0.71751 0.8696	0.56478 0.7035	0.17352 0.2687	-0.30426 -0.2687	-0.67105 -0.7035	-0.77030 -0.8696	-0.55803 -0.7035
0.7	0.68345 0.8511	0.48512 0.6325	0.00193 0.0089	-0.50438 -0.5002	-0.75908 -0.8325	-0.60690 -0.7370	-0.12557 -0.2630
0.8	0.65207 0.8333	0.40494 0.5576	-0.15764 -0.0871	-0.64839 -0.6742	-0.70794 -0.8151	-0.28196 -0.4167	-0.34837 0.2575
0.9	0.62311 0.8163	0.32489 0.4798	-0.30342 -0.2523	-0.71705 -0.7764	-0.53357 -0.6604	-0.11292 0	0.68888 0.6604
1	0.59631 0.8000	0.24646 0.4000	-0.42971 -0.4000	-0.71236 -0.8000	-0.27088 -0.4000	0.47168 0.4000	0.77083 0.8000
1.1					0.03234 -0.0820	0.70195 0.6792	0.57176 0.6345
1.2	0.54871 0.7692	0.09080 0.2377	-0.61128 -0.6223	-0.49212 -0.6223	0.32351 0.2377	0.74844 0.7692	0.17692 0.2377
1.3					0.55552 0.5050	0.60510 0.6536	-0.25679 -0.2332
1.4	0.50774 0.7407	-0.05417 0.0774	-1.66604 -0.7246	-0.10096 -0.2289	0.69151 0.6767	0.31656 0.3704	-0.56877 -0.5993
1.5					0.71102 0.7273	-0.03694 0	-0.64864 -0.7273
1.6	0.47222 0.7143	-0.18277 -0.0747	-0.60061 -0.6987	0.31108 0.2207	0.61422 0.6525	-0.36090 -0.3571	-0.47375 -0.5779
1.7					0.42112 0.4696	-0.57185 -0.6077	-0.11764 -0.2168
1.8	0.44121 0.6896	-0.29399 -0.2131	-0.43570 -0.5579	0.59482 0.5579	0.16725 0.2131	-0.61889 -0.6897	0.27827 0.2131
1.9					-0.10209 -0.0709	-0.49630 -0.5871	0.56262 0.5485

TABLE VIII (Cont.)

z \ n	0	1	2	3	4	5	6
2	0.41382 0.6667	-0.38564 -0.3333	-0.20577 -0.3333	0.65169 0.6667	-0.34021 -0.3333	-0.24208 -0.3333	0.63398 0.6667
2.1					-0.50726 -0.5303	0.07488 0	0.47585 0.5305
2.2	0.38990 0.6452	-0.45277 -0.4317	0.04882 -0.0674	0.47582 0.5219	-0.58040 -0.6311	0.36602 0.3226	0.14930 0.1994
2.3					-0.54948 -0.6210	0.55682 0.5497	-0.21735 -0.1962
2.4	0.36847 0.6250	-0.49865 -0.5056	0.28317 0.1931	0.14351 0.1931	-0.42308 -0.5056	0.60008 0.6250	-0.48367 -0.5056
2.5					-0.22528 -0.3077	0.48863 0.5329	-0.55314 -0.6154
2.6	0.34907 0.6061	-0.51800 -0.5537	0.46201 0.4055	-0.21180 -0.1872	0.00853 -0.0634	0.25721 0.3030	-0.40556 -0.4903
2.7					0.23860 0.1845	-0.03202 0	-0.10716 -0.1845
2.8	0.33210 0.5882	-0.51485 -0.5754	0.55910 0.5374	-0.46033 -0.4759	0.42392 0.3936	-0.29910 -0.2941	0.23067 0.1818
2.9					0.53490 0.5296	-0.47539 -0.5020	0.47823 0.4690
3	0.31661 0.5714	-0.48915 -0.5714	0.56210 0.5714	-0.51861 -0.5714	0.55443 0.5714	-0.51702 -0.5714	0.54413 0.5714

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_n = \frac{B}{U_\infty} \frac{\partial \phi_n}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE COSINE FORMULATION, FOR WHICH $\eta_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$

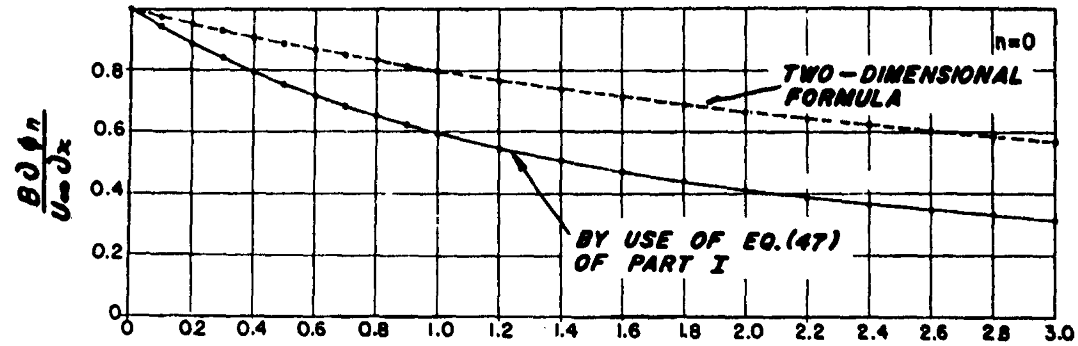

FIGURE 3a

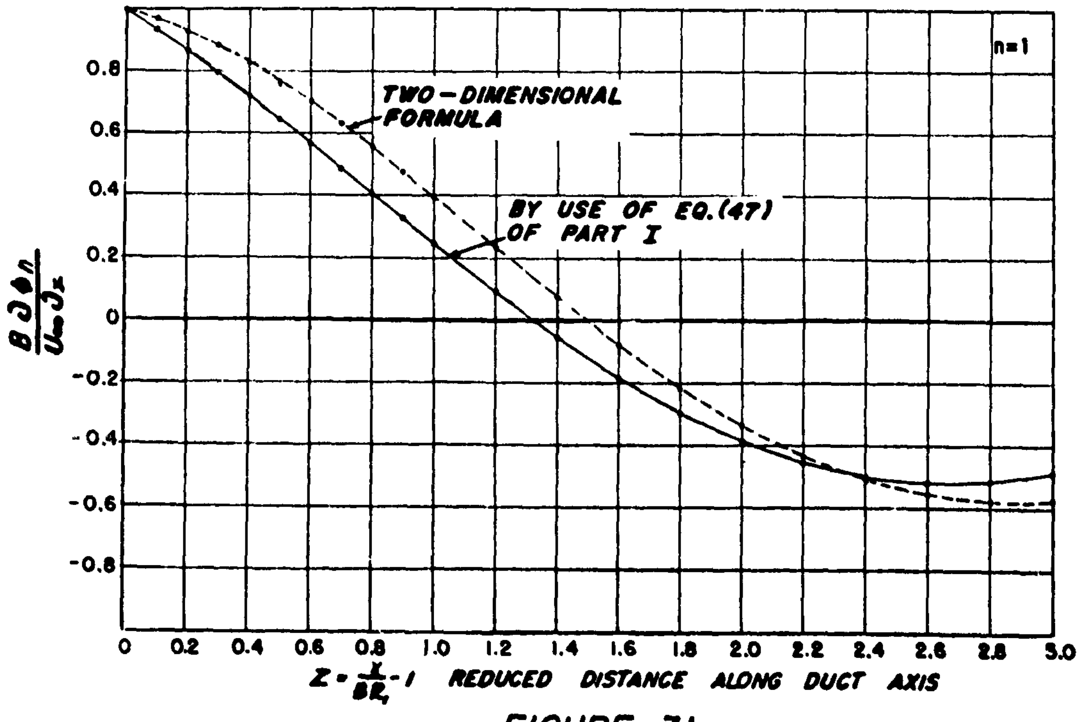

FIGURE 3b

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_n = \frac{B}{U_\infty}\frac{\partial \phi_n}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE COSINE FORMULATION, FOR WHICH $\eta_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$

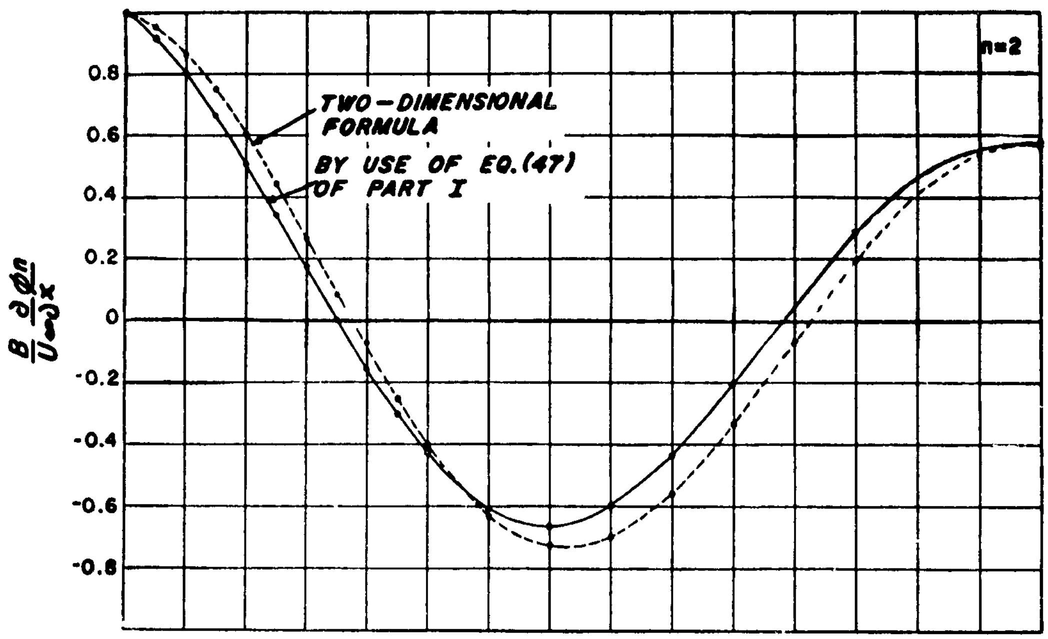

FIGURE 3c

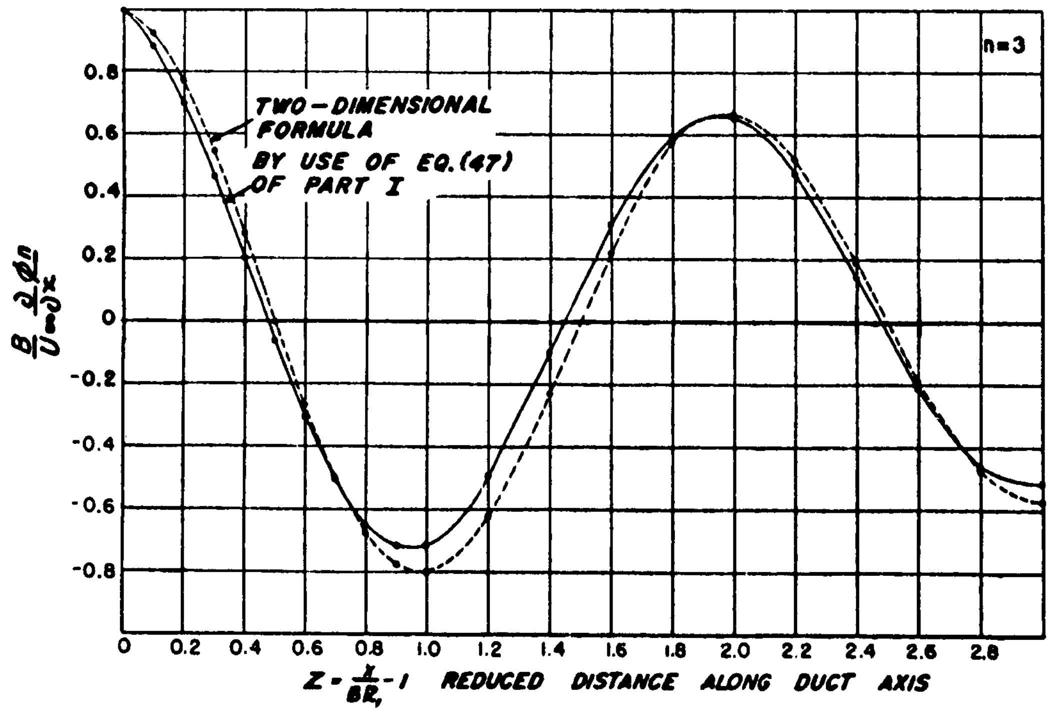

FIGURE 3d

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_n = \frac{B}{U_\infty} \frac{\partial \phi_n}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_n = \frac{\cos n\theta}{1+\beta B(l_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE COSINE FORMULATION, FOR WHICH $\eta_n = \frac{\cos n\theta}{1+\beta B(l_1/\pi)\theta}$

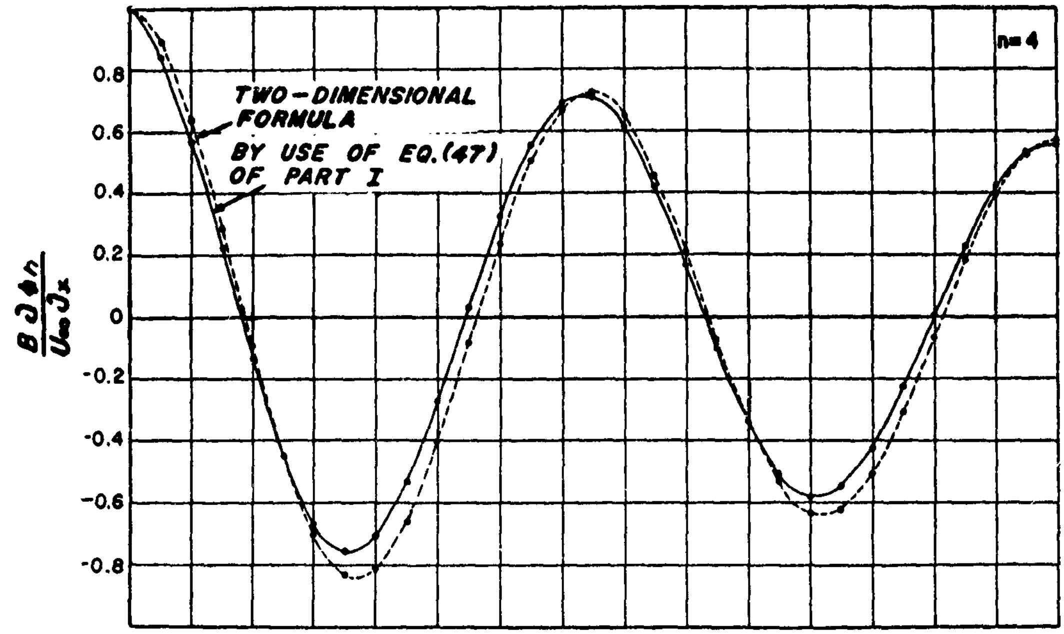

FIGURE 3e

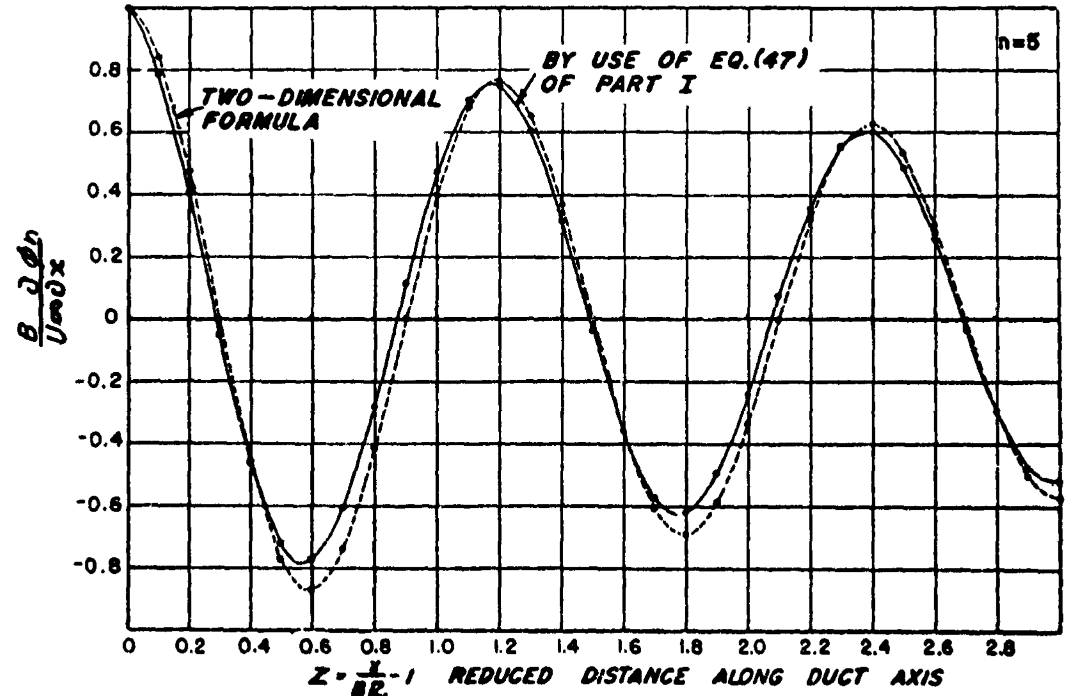

FIGURE 3f

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_n = \frac{B}{U_\infty} \frac{\partial \phi_n}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE COSINE FORMULATION, FOR WHICH $\eta_n = \frac{\cos n\theta}{1+\beta B(\ell_1/\pi)\theta}$

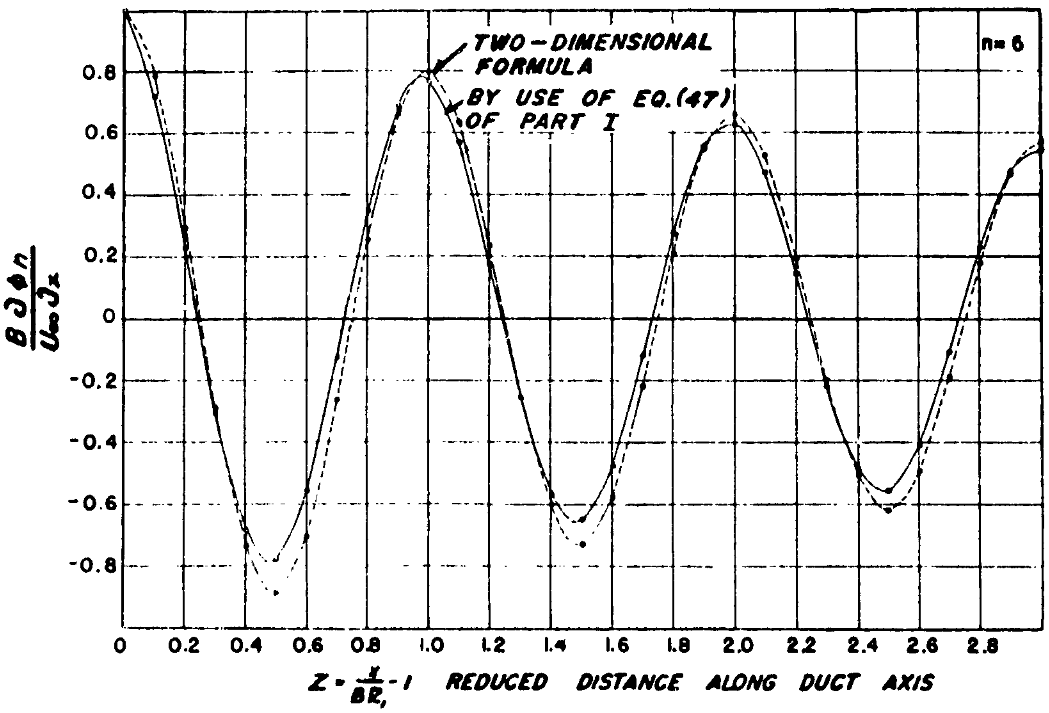

FIGURE 3g

are displayed graphically in Fig. 3, for the seven harmonic components being employed throughout these calculations.

The values of $+\frac{B}{U_\infty}\frac{\partial \phi_n}{\partial x}$ have also been computed on the basis of Ackeret's two-dimensional formula for each one of the radial components, η_n. These values are simply given as $F_n^* = \frac{\cos n\theta}{1+\beta B \frac{l}{\pi}\theta}$, and these two-dimensional results have been entered into Table VIII likewise, and, in addition, they have also been plotted for comparative purposes in Fig. 3.

In the corresponding calculations which have to be made when it comes to use of the sine formulations for the radial components, no difficulty such as encountered above will arise, because in the process of carrying out the numerical evaluations in the case where $\eta_{m+k} = \frac{\sin k\theta}{1+\beta B \frac{l}{\pi}\theta}$ it will be found that the integral that defines $\frac{\partial \phi_k}{\partial x}$, in the expression for

$$-F_k(z) = -\frac{B}{U_\infty}\frac{\partial \phi_k}{\partial x}$$

has no singularity, in this case, at the point where z = s = 0. The straight-forward numerical computations have been easily accomplished for the $F_k(z)$ functions, consequently, and the results are presented in Table IX, while the corresponding graphical plots are given in Fig. 4. The values of F_k^*, which are defined according to the relation

$$F_k^* = \frac{\sin k\theta}{1+\beta B \frac{l}{\pi}\theta}$$

corresponding to the two-dimensional interpretation of the problem that makes use of Ackeret's formula for the pressures, have also been computed, and these results have likewise been entered into Table IX and plotted up on the graphs of Fig. 4, for comparison with the more exact three-dimensional values.

TABLE IX

AXIAL COMPONENTS OF INDUCED VELOCITIES AT THE DUCT'S SURFACE, $F_k = +\frac{B}{U_\infty}\frac{\partial \phi}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I, AND COMPARISON OF THESE VALUES WITH THOSE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_k^* = \frac{\sin k\,\theta}{1+\beta B\frac{l_1}{\pi}\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE SINE FORMULATION FOR WHICH $\eta_{m+k} = \frac{\sin k\,\theta}{1+\beta B\frac{l_1}{\pi}\theta}$

z \ k	1	2	3	4	5	6
0	$F_1 = 0$ $F_1^* = 0$	$F_2 = 0$ $F_2^* = 0$	$F_3 = 0$ $F_3^* = 0$	$F_4 = 0$ $F_4^* = 0$	$F_5 = 0$ $F_5^* = 0$	$F_6 = 0$ $F_6^* = 0$
0.1	0.09101 0.1020	0.18066 0.2028	0.26724 0.3015	0.34907 0.3968	0.42470 0.4878	0.49289 0.5734
0.2	0.18084 0.1980	0.35233 0.3874	0.50502 0.5598	0.63032 0.7078	0.72005 0.8248	0.77009 0.9058
0.3	0.26033 0.2875	0.49191 0.5468	0.66840 0.7526	0.76681 0.8847	0.78164 0.9302	0.70584 0.8847
0.4	0.33038 0.3698	0.59767 0.6756	0.74732 0.8646	0.75269 0.9041	0.61234 0.7873	0.34851 0.5343
0.5	0.39137 0.4444	0.66697 0.7698	0.74423 0.8889	0.59885 0.7698	0.27015 0.4444	−0.15203 0
0.6	0.44340 0.5111	0.69938 0.8270	0.66220 0.8270	0.33841 0.5111	−0.14594 0	−0.59830 −0.5111
0.7	0.48657 0.5695	0.69835 0.8464	0.51221 0.6885	0.01980 0.1769	−0.52274 −0.4255	−0.88395 −0.8094
0.8	0.52105 0.6193	0.66228 0.8288	0.31335 0.4898	−0.30007 −0.1732	−0.76216 −0.7217	−0.75262 −0.7925
0.9	0.54700 0.6604	0.59652 0.7764	0.08539 0.2523	−0.56649 −0.4798	−0.80606 −0.8163	−0.42121 −0.4798
1	0.56412 0.6928	0.50405 0.6228	−0.14830 0	−0.73607 −0.6928	−0.64979 −0.6928	0.03864 0
1.1				−0.78330 −0.7800	−0.34093 −0.3921	0.45175 0.4610
1.2	0.57443 0.7316	0.25538 0.4521	−0.54885 −0.4521	−0.70417 −0.7316	0.03494 0	0.66659 0.7316
1.3				−0.51640 −0.5609	0.37799 0.3773	0.61183 0.7178
1.4	0.55860 0.7367	−0.02457 0.1540	−0.73017 −0.7045	−0.25508 −0.3013	0.60048 0.6415	0.31922 0.4354
1.5				0.03359 0	0.64933 0.7273	−0.09252 0
1.6	0.51381 0.7104	−0.29319 −0.1485	−0.65010 −0.6793	0.30060 0.2905	0.51862 0.6186	−0.46496 −0.4198
1.7				0.50223 0.5215	0.24936 0.3509	−0.65924 −0.6674

TABLE IX (Cont.)

z \ k	1	2	3	4	5	6
1.8	0.44713 0.6559	−0.50572 −0.4054	−0.35261 −0.4054	0.60715 0.6559	−0.08265 0	−0.60458 −0.6559
1.9				0.60259 0.6742	−0.38622 −0.3390	−0.33436 −0.3985
2	0.36303 0.5773	−0.62876 −0.5773	0.04410 0	0.49100 0.5773	−0.58494 −0.5773	−0.04483 0
2.1				0.29433 0.3854	−0.62984 −0.6557	0.38827 0.3854
2.2	0.26438 0.4794	−0.65014 −0.6416	0.38613 0.3792	0.05173 0.1341	−0.51123 −0.5587	0.57098 0.6136
2.3				−0.19558 −0.1320	−0.26648 −0.3175	0.52802 0.6038
2.4	0.15833 0.3674	−0.56822 −0.5944	0.55742 0.5944	−0.40527 −0.3674	0.03639 0	0.28297 0.3674
2.5				−0.54283 −0.5329	0.31626 0.3077	−0.06570 0
2.6	0.04837 0.2465	−0.40303 −0.4504	0.50863 0.5764	−0.58884 −0.6027	0.49926 0.5249	−0.38338 −0.3562
2.7				−0.53952 −0.5678	0.54374 0.5970	−0.55262 −0.5678
2.8	−0.05999 0.1223	−0.18585 −0.2392	0.26589 0.3457	−0.39336 −0.4371	0.43892 0.5094	−0.51525 −0.5594
2.9				−0.19293 −0.2358	0.21816 0.2898	−0.28595 −0.3407
3	−0.16295 0	0.04540 0	−0.06629 0	0.03206 0	−0.05697 0	0.03871 0

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_k = \frac{B}{U_\infty}\frac{\partial \phi_k}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_k = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE SINE FORMULATION, FOR WHICH $\eta_{\pi+k} = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$

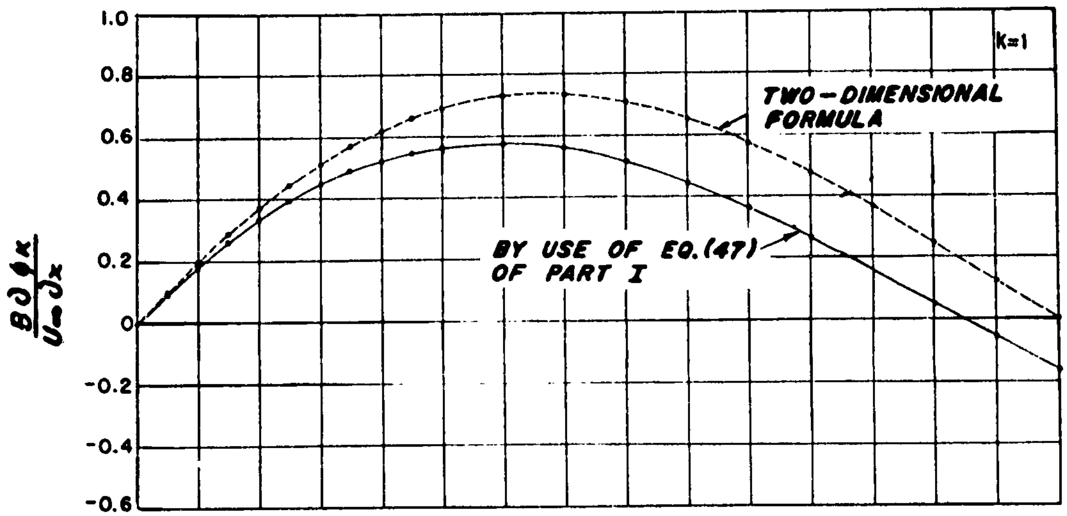

FIGURE 4a

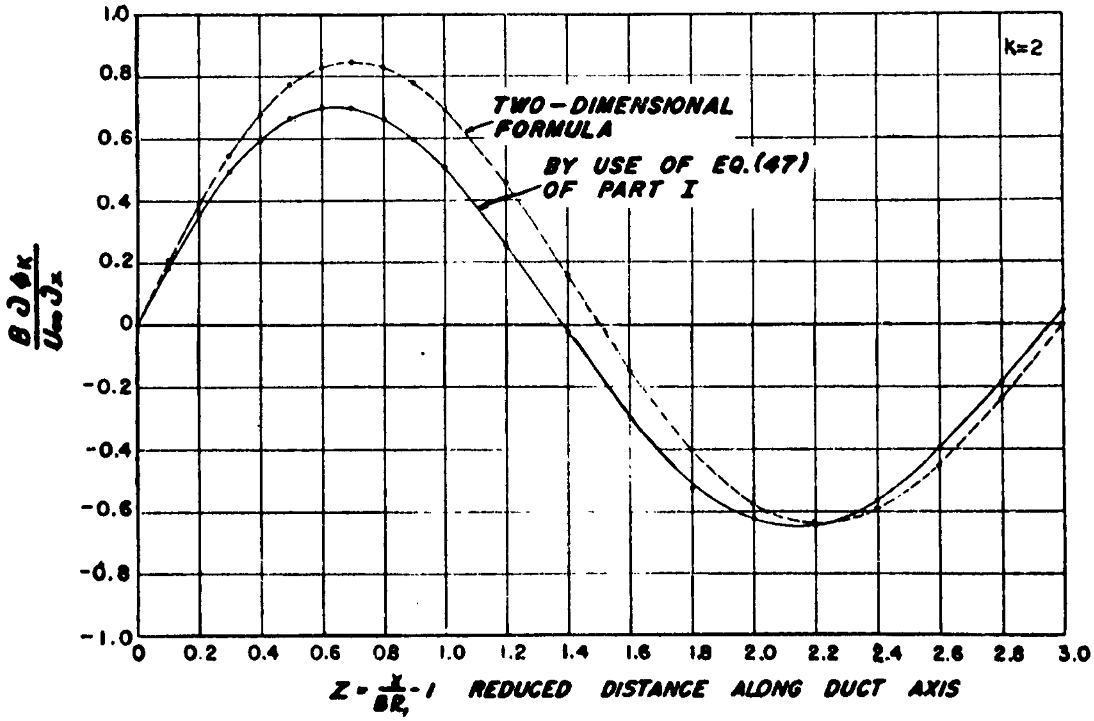

FIGURE 4b

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_k = \frac{B}{U_\infty} \frac{\partial \phi_k}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_k = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE SINE FORMULATION, FOR WHICH $\eta_{m+k} = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$

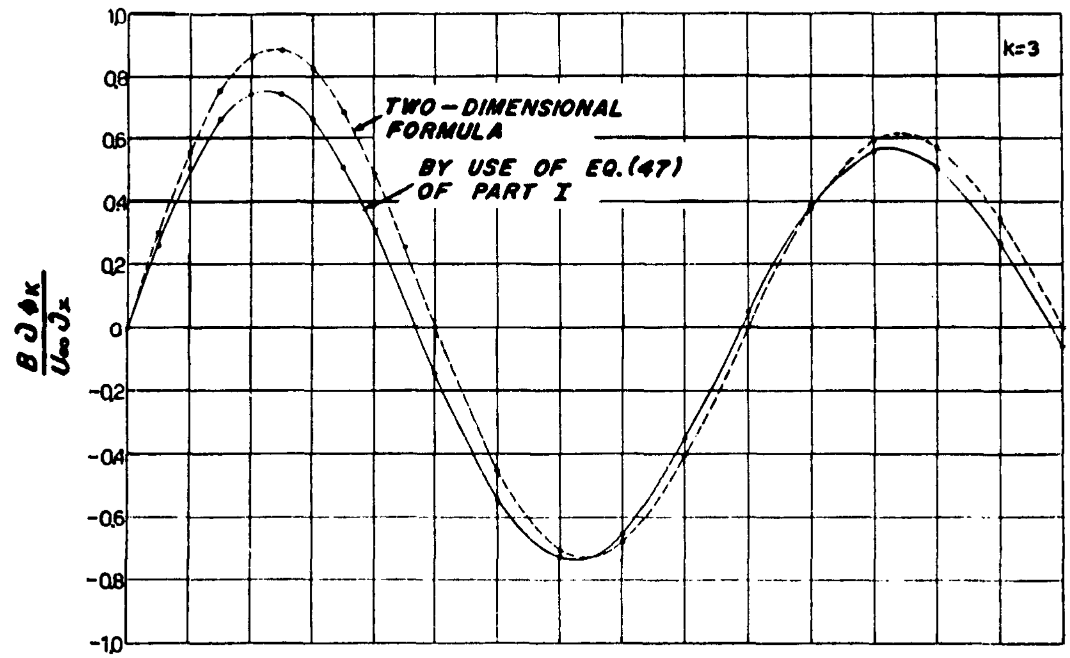

FIGURE 4c

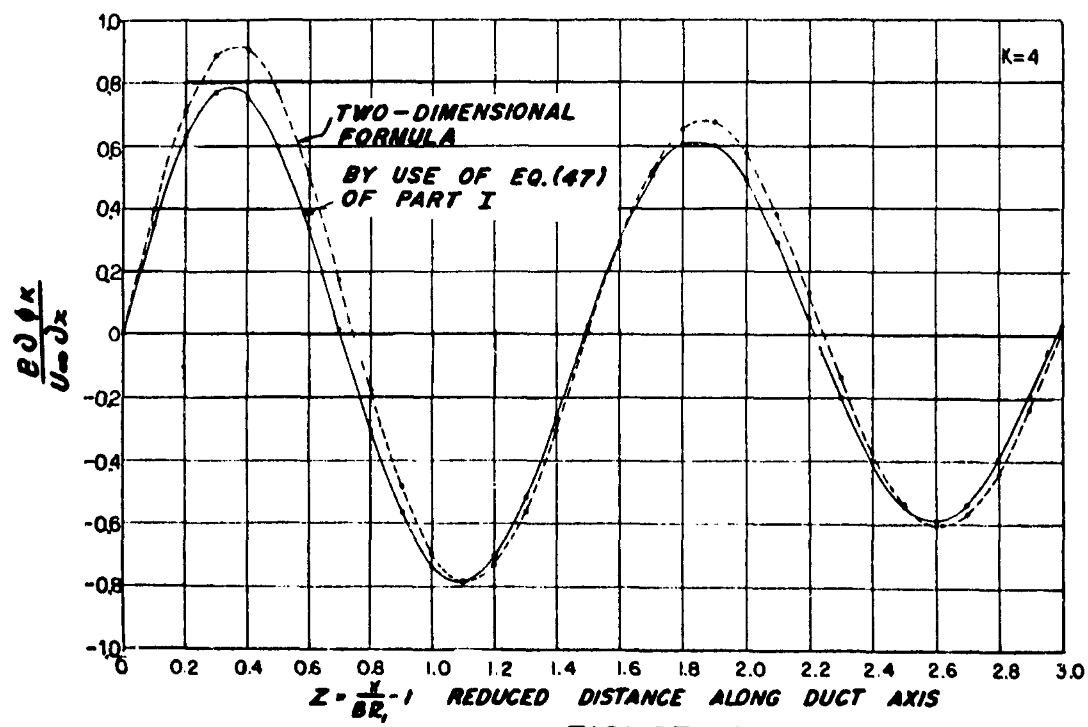

FIGURE 4d

AXIAL COMPONENTS OF INDUCED VELOCITY AT THE DUCT'S SURFACE, $F_k = \frac{B}{U_\infty} \frac{\partial \phi_k}{\partial x}$, CALCULATED BY MEANS OF EQ. (47) OF PART I AND COMPARISON OF THIS CURVE WITH THE ONE OBTAINED FROM THE PROCEDURE EMPLOYING ACKERET'S TWO-DIMENSIONAL FORMULA, IN WHICH CASE $F_k = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$, WHEN THE RADIAL COMPONENTS ARE DESCRIBED BY MEANS OF THE SINE FORMULATION, FOR WHICH $\eta_{m+k} = \frac{\sin k\theta}{1+\beta B(\ell_1/\pi)\theta}$

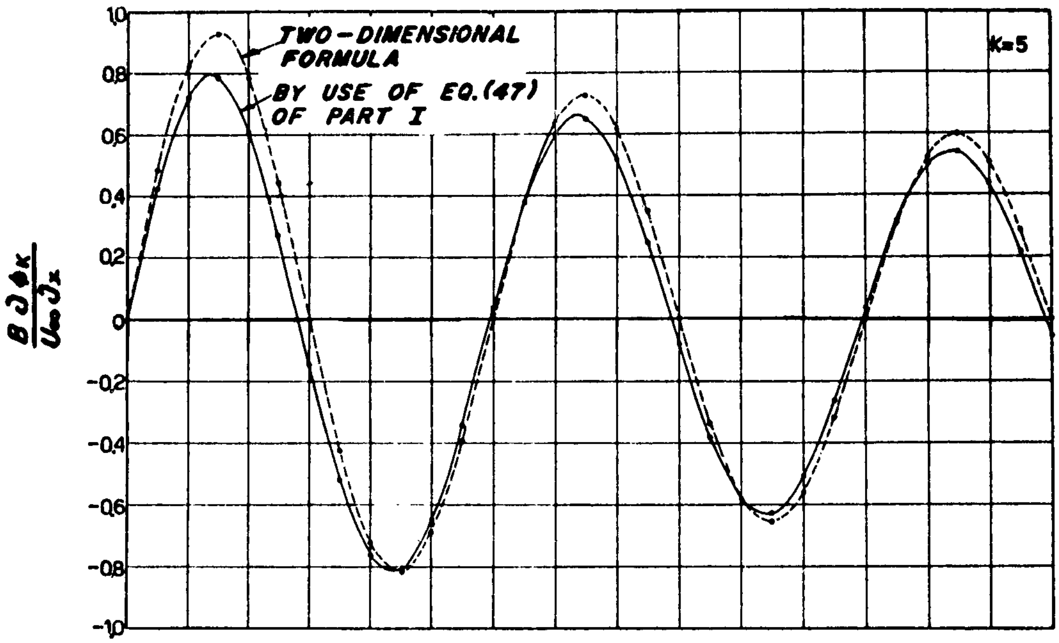

FIGURE 4e

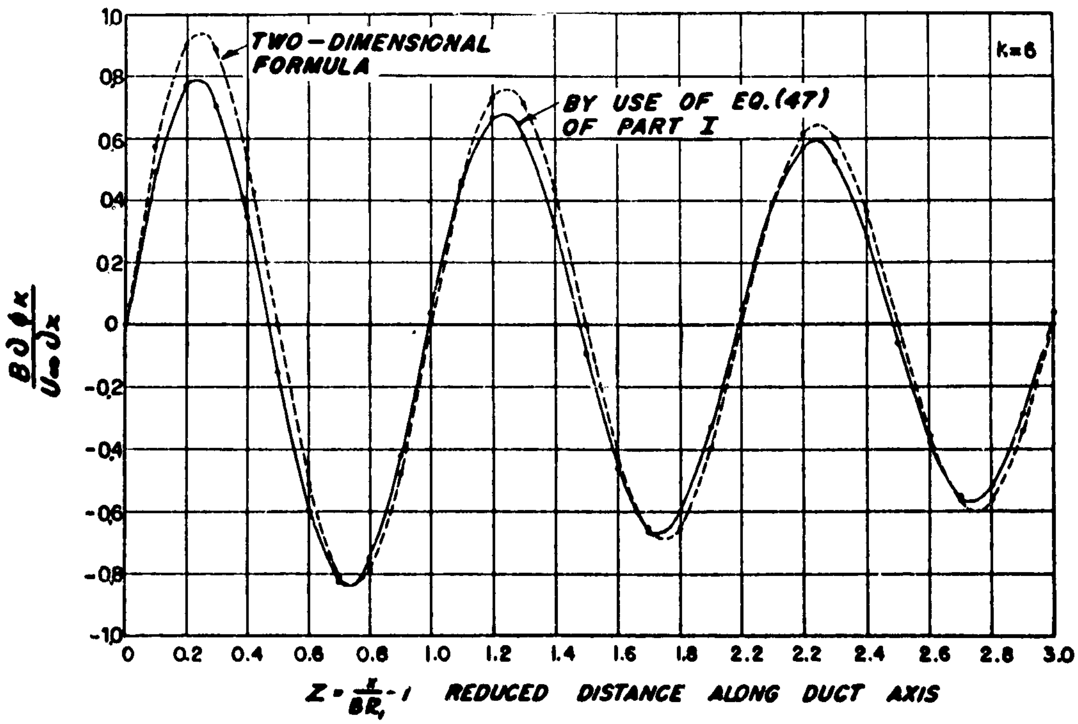

FIGURE 4f

It may immediately be seen upon comparison of the F_n values with the F_n^* values, and of the F_k values with the F_k^* values that, throughout the whole z-range extending from 0 to 3.0, these values do not differ from one another to any appreciable degree except for the smallest values of n and k.

It is worthwhile to point out that, in the case where $\beta = 0$, the corresponding F and F^* values will become equal as n and k become infinite. On the basis of this observation one may cautiously presume that the same result will hold true even when the value of β is not zero, though still of small size; but in the cases which come up in most practical applications it will be adequate, in general to consider merely such suitably small values of β.

To make the above statement clear in the case where $\beta = 0$, one may proceed to determine formally the field of flow which is external to an infinitely long circular cylinder, by use of the three-dimensional approach already illustrated. In this case the axis of the cylinder is taken to be along the x-axis and its radius is to be equal to $B\,R_o$, so that the radial coordinate is defined as $r = \frac{R}{B\,R_o}$.

The n'th one of the harmonic components defining the radial velocity produced by action of the corresponding supersonic source distribution, which is arrayed along the axis of the cylinder in question, has the magnitude denoted by $V_\infty \cos(nz)$ at the surface (i.e., at all points ranged along the generatrix of the cylinder lying in any one of the arbitrary general meridional planes passing through its axis). The velocity potential which describes such a flow produced by the proper distribution of supersonic sources placed along the axis is given by means of the formula

$$\phi_n = U_\infty \varphi_n^* = U_\infty \left[A\left(Y_0 \cos(nz) - J_0 \sin(nz)\right) + C\left(J_0 \cos(nz) + Y_0 \sin(nz)\right)\right]$$

where J_0 and Y_0 are the cylindrical Bessel functions of the first and second kinds, and depending on the parameter nBr, while the symbols A and C stand for constants which are determined by the boundary condition. This boundary condition to be satisfied is merely:

$$\left(\frac{\partial \varphi_n^*}{\partial r}\right)_{r=1} = \cos(nz) \ .$$

It is found, upon applying this condition, that

$$C = \frac{J_0'(B_n)}{Y_0'(B_n)} A \qquad \text{and} \qquad A = \frac{1}{nB} \cdot \frac{Y_0'(nB)}{Y_0'^2(nB) + J_0'^2(nB)} \ .$$

It follows, consequently, that the proper expression for the axial component of the auxiliary potential is given by

$$\left(\frac{\partial \varphi_n^*}{\partial z}\right)_{r=1} = -\frac{1}{B} \frac{Y_0 Y_0' + J_0 J_0'}{Y_0'^2 + J_0'^2} \sin(nz) + \frac{1}{B} \frac{Y_0 J_0' - J_0 Y_0'}{Y_0'^2 + J_0'^2} \cos(nz)$$

at the surface of the cylinder.

But if the limiting values that these coefficients of sin(nz) and cos (nz) take on as $n \longrightarrow \infty$ are examined it is found that

$$\lim_{n\to\infty} \frac{Y_0 Y_0' + J_0 J_0'}{Y_0'^2 + J_0'^2} = -\lim_{n\to\infty} \frac{\dfrac{\cos\varphi_0 \cos\varphi_1 + \sin\varphi_0 \sin\varphi_1}{\frac{1}{2}\pi n B}}{\dfrac{1}{\frac{1}{2}\pi n B}}$$

$$= -\cos(\varphi_1 - \varphi_0) = 0$$

because, as $n \to \infty$ it is true that

$$J_0(nB) \cong \frac{\cos \varphi_0}{\sqrt{\frac{1}{2}\pi n B}}$$

$$Y_0(nB) \cong \frac{\sin \varphi_0}{\sqrt{\frac{1}{2}\pi n B}}$$

$$J_1(nB) \cong \frac{\cos \varphi_1}{\sqrt{\frac{1}{2}\pi n B}}$$

and
$$Y_1(nB) \cong \frac{\sin \varphi_1}{\sqrt{\frac{1}{2}\pi n B}}$$

wherein $\varphi_0 = nB - \frac{\pi}{4}$ and $\varphi_1 = nB - (1 + \frac{1}{2})\frac{\pi}{2}$.

In like manner it may also be seen that

$$\lim_{n \to \infty} \frac{Y_0 J_0' - J_0 Y_0'}{Y_0'^2 + J_0'^2} = \sin(\varphi_1 - \varphi_0) = 1$$

and thus, for sufficiently large values of n, it follows that

$$\left(\frac{\partial \varphi_n^*}{\partial \gamma}\right)_{r=1} \cong \frac{1}{B} \cos n\gamma$$

which is identical with the result that one comes to in the two-dimensional treatment through use of the Ackeret formula for relating the pressure to the geometrical characteristics of the duct.

5. Evaluation of the Constants A_m and A_{m+k}

Having obtained the values of F_n, F_n^*, F_k, and F_k^*, the next step in the procedure described in Part I is to determine the quantities ε_m and ε_{m+k}. In the particular example under consideration the appropriate formulae for carrying out the determinations are[†]

$$\left.\begin{aligned} \varepsilon_n &= \frac{2\int_0^3 (F_n - F_n^*)\cos\left(\frac{n\pi}{3}\gamma\right)d\gamma}{\frac{\ell_1^2}{\pi^2}} = 2.19325\int_0^3 (F_n - F_n^*)\cos\left(\frac{n\pi}{3}\gamma\right)d\gamma \\ \text{and } \varepsilon_{\sigma+k} &= \frac{2\int_0^3 (F_k - F_k^*)\sin\left(\frac{k\pi}{3}\gamma\right)d\gamma}{\frac{\ell_1^2}{\pi^2}} = 2.19325\int_0^3 (F_k - F_k^*)\sin\left(\frac{k\pi}{3}\gamma\right)d\gamma \end{aligned}\right\} \quad (10)$$

[†] It should be noted that in the numerical evaluation of the ε-values one needs to compute the integral $\int_0^\pi \frac{\cos^2 n\theta\, d\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}$ as well as the integral $\int_0^\pi \frac{\sin^2 n\theta\, d\theta}{1+\beta B \frac{\ell_1}{\pi}\theta}$, and these computations can, if desired, be carried out by means of the procedure described in Part I. As a practical way of handling this computational chore, however, it will be more convenient to proceed by evaluation of the integrals written on the right hand side of the Eqs. (10). As a matter of fact, the ε-values actually arise as the difference between two quantities which differ but slightly from one another. If the integrals of the F_n functions were to be carried out by the approximate method (and this approximation will naturally become less accurate as n grows to large values) and if the integrals of the F_n^* functions were to be obtained exactly, then the resulting error which would be introduced into the corresponding values for ε would actually be larger than the errors which are made in carrying out the integrations on the $F_n - F_n^*$ differences by just numerical means.

Such computations have been made and the results are as follows for the specific example under consideration:

$\varepsilon_0 = 1.34424$	$\varepsilon_7 = 0.67938$
$\varepsilon_1 = 0.25518$	$\varepsilon_8 = 0.27054$
$\varepsilon_2 = 0.17318$	$\varepsilon_9 = 0.23286$
$\varepsilon_3 = 0.16114$	$\varepsilon_{10} = 0.19772$
$\varepsilon_4 = 0.15921$	$\varepsilon_{11} = 0.18$
$\varepsilon_5 = 0.15728$	$\varepsilon_{12} = 0.17$
$\varepsilon_6 = 0.15535$	

Now the coefficients $\mu_{n,i}$ which are employed in Eqs. (59') and (59") of Part I have the values which are displayed in the appended Table X, so that the equations which are used in determining the sought A_n values take the form:

$$\varepsilon_0 = 1.34424$$

$$= \mu_{0,0}A_0 + \mu_{0,7}\left(\frac{\varepsilon_7}{\mu_{7,7}} - \frac{\mu_{7,0}}{\mu_{7,7}}A_0 - \frac{\mu_{7,2}}{\mu_{7,7}}A_2 - \frac{\mu_{7,4}}{\mu_{7,7}}A_4 - \frac{\mu_{7,6}}{\mu_{7,7}}A_6\right)$$

$$+ \mu_{0,9}\left(\frac{\varepsilon_9}{\mu_{9,9}} - \frac{\mu_{9,0}}{\mu_{9,9}}A_0 - \frac{\mu_{9,2}}{\mu_{9,9}}A_2 - \frac{\mu_{9,4}}{\mu_{9,9}}A_4 - \frac{\mu_{9,6}}{\mu_{9,9}}A_6\right)$$

$$+ \mu_{0,11}\left(\frac{\varepsilon_{11}}{\mu_{11,11}} - \frac{\mu_{11,0}}{\mu_{11,11}}A_0 - \frac{\mu_{11,2}}{\mu_{11,11}}A_2 - \frac{\mu_{11,4}}{\mu_{11,11}}A_4 - \frac{\mu_{11,6}}{\mu_{11,11}}A_6\right)$$

$$= 3.6577A_0 - 0.82166A_2 - 0.15299A_4 - 0.06183A_6 + (1.62114)(0.67938)$$

$$+ (0.06004)(0.23286) + (0.06485)(0.18).$$

$$\varepsilon_2 = 0.17318$$

$$= \mu_{2,2}A_2 + \mu_{2,7}\left(\frac{\varepsilon_7}{\mu_{7,7}} - \frac{\mu_{7,0}}{\mu_{7,7}}A_0 - \frac{\mu_{7,2}}{\mu_{7,7}}A_2 - \frac{\mu_{7,4}}{\mu_{7,7}}A_4 - \frac{\mu_{7,6}}{\mu_{7,7}}A_6\right)$$

TABLE X

ARRAY OF COEFFICIENTS, $\mu_{n,i}$, APPEARING IN EQUATIONS DETERMINING ε's, ACCORDING TO EQ.(59) OF PART I, AND THEMSELVES DEFINED IN EQ.(58) OF PART I

1st Subscript (n) \ 2nd Subscript (i)	0	1	2	3	4	5	6	7	8	9	10	11	12
0	9.86960	0	0	0	0	0	0	4	0	0.14815	0	0.16000	0
1	0	2.46740	0	0	0	0	0	0	1.77778	0	0.28444	0	0.11755
2	0	0	2.46740	0	0	0	0	0.44444	0	1.44000	0	0.22676	0
3	0	0	0	2.46740	0	0	0	0	0.64000	0	1.30612	0	0.19753
4	0	0	0	0	2.46740	0	0	0.01778	0	0.73469	0	1.23457	0
5	0	0	0	0	0	2.46740	0	0	0.03628	0	0.79012	0	1.19008
6	0	0	0	0	0	0	2.46740	0.00326	0	0.04938	0	0.82645	0
7	4	0	0.44444	0	0.01778	0	0.00326	2.46740	0	0	0	0	0
8	0	1.77778	0	0.64000	0	0.03628	0	0	2.46740	0	0	0	0
9	0.14815	0	1.44000	0	0.73469	0	0.04938	0	0	2.46740	0	0	0
10	0	0.28444	0	1.30612	0	0.79012	0	0	0	0	2.46740	0	0
11	0.16000	0	0.22676	0	1.23457	0	0.82645	0	0	0	0	2.46740	0
12	0	0.11755	0	0.19753	0	1.19008	0	0	0	0	0	0	2.46740

$$+\mu_{2,9}\left(\frac{\varepsilon_9}{\mu_{9,9}}-\frac{\mu_{9,0}}{\mu_{9,9}}A_0-\frac{\mu_{9,2}}{\mu_{9,9}}A_2-\frac{\mu_{9,4}}{\mu_{9,9}}A_4-\frac{\mu_{9,6}}{\mu_{9,9}}A_6\right)$$

$$+\mu_{2,11}\left(\frac{\varepsilon_{11}}{\mu_{11,11}}-\frac{\mu_{11,0}}{\mu_{11,11}}A_0-\frac{\mu_{11,2}}{\mu_{11,11}}A_2-\frac{\mu_{11,4}}{\mu_{11,11}}A_4-\frac{\mu_{11,6}}{\mu_{11,11}}A_6\right)$$

$= -0.82164A_0+1.52611A_2-0.54543A_4-0.10536A_6+(0.18012)(0.67938)$

$+(0.58361)(0.23286)+(0.09190)(0.18).$

$\varepsilon_4= 0.15921$

$= -0.15301\ A_0-0.54543A_2+1.63079A_4-0.42823A_6+(0.00721)(0.67938)$

$+(0.29776)(0.23286)+(0.50035)(0.18).$

$\varepsilon_6= 0.15535$

$= -0.06183A_0-0.10535A_2-0.42824A_4+2.18959A_6+(0.00132)(0.67938)$

$+(0.02001)(0.23286)+(0.33495)(0.18).$

$\varepsilon_1= 0.25518$

$= 1.14810A_1-0.62111A_3-0.17392A_5+(0.72051)(0.27054)$

$(0.11528)(0.19772)+(0.04764)(0.17).$

$\varepsilon_3= 0.16114$

$= -0.62110A_1+1.59420A_3-0.52294A_5+(0.25938)(0.27054)$

$+(0.52935)(0.19772)+(0.08006)(0.17).$

$\varepsilon_5= 0.15728$

$= -0.17391A_1-0.52293A_3+1.63986A_5+(0.01470)(0.27054)$

$+(0.32022)(0.19772)+(0.048232)(0.17).$

These ecuations then reduce down into the simple forms

$$\left.\begin{aligned}
3.36577A_0-0.82166A_2-0.15299A_4-0.06183A_6 &= 0.21722\\
0.82164A_0-1.52611A_2+0.54543A_4+0.10536A_6 &= 0.10163\\
0.15301A_0+0.54543A_2-1.63079A_4+0.42823A_6 &= 0.00509\\
0.06183A_0+0.10535A_2+0.42824A_4-2.18959A_6 &= -0.08950
\end{aligned}\right\}$$

$$\left.\begin{aligned}
1.14810A_1-0.62111A_3-0.17392A_5 &= 0.02936\\
0.62110A_1-1.59420A_3+0.52294A_5 &= 0.02731\\
0.17391A_1+0.52293A_3-1.63986A_5 &= -0.00799
\end{aligned}\right\}$$

These two sets of equations may be solved for the A's, to give

$A_0 = 0.057659$	and	$A_1 = 0.022833$
$A_2 = -0.031776$		$A_3 = -0.006525$
$A_4 = 0.002552$		$A_5 = 0.005213$
$A_6 = 0.041474$		

Likewise, the values for the A_{m+k} coefficients are obtainable from the relations

$$A_7 = \frac{1}{\mu_{7,7}}\left(\varepsilon_7-\mu_{7,0}A_0-\mu_{7,2}A_2-\mu_{7,4}A_4-\mu_{7,6}A_6\right)$$

$$A_9 = \frac{1}{\mu_{9,9}}\left(\varepsilon_9-\mu_{9,0}A_0-\mu_{9,2}A_2-\mu_{9,4}A_4-\mu_{9,6}A_6\right)$$

$$A_{11} = \frac{1}{\mu_{11,11}}\left(\varepsilon_{11}-\mu_{11,0}A_0-\mu_{11,2}A_2-\mu_{11,4}A_4-\mu_{11,6}A_6\right)$$

$$A_8 = \frac{1}{\mu_{8,8}}\left(\varepsilon_8-\mu_{8,1}A_1-\mu_{8,3}A_3-\mu_{8,5}A_5\right)$$

$$A_{10} = \frac{1}{\mu_{10,10}}\left(\varepsilon_{10}-\mu_{10,1}A_1-\mu_{10,3}A_3-\mu_{10,5}A_5\right)$$

$$A_{12} = \frac{1}{\mu_{12,12}}\left(\varepsilon_{12}-\mu_{12,1}A_1-\mu_{12,3}A_3-\mu_{12,5}A_5\right)$$

so that $A_7 = 0.187536$	and	$A_8 = 0.110043$
$A_9 = 0.107867$		$A_{10} = 0.079285$
$A_{11} = 0.056964$		$A_{12} = 0.065819$.

It is worthwhile to make the following observation about these results at this juncture. Let these just-determined values of A_n and A_{m+k} be

inserted into Eq. (57) of Part I in order to find the corresponding value of S^{**}. Now if the so-determined S^{**} function is used in Eq.(48) of Part I, then, because of the physical significance attached to the ε quantities that was pointed out in the theoretical development, it follows that the <u>exact</u> values of the drag coefficient will be obtained which pertains to a duct, the radial-velocity-component parameter of which is defined by

$$\eta(z)\left(1+\beta B \frac{\ell_1}{\pi}\theta\right)$$

and which is represented by a trigonometric series in cos(nθ) and sin (nθ), with n taking on integral values from 0 to six in the particular example now under examination.

DETERMINATION OF THE BEST DUCT CONTOUR UNDER CONDITION (a) TOGETHER WITH CONDITION (b) OR (c)

6. Calculation of the Coefficients H_n and K_k

The values of the coefficients H_n and K_k can be determined by use of Ecs. (65) of Part I, once the coefficients B_n and C_n appearing therein have been evaluated. This evaluation for the B_n and C_n coefficients is carried out by having recourse to Ecs. (64) of Part I. Making use of the specific date valid for this example, it is found that

$B_o = 3.14159\,\lambda_1 + 4.71239\,\lambda_2$ $\qquad$ $B_4 = 0$

$B_1 = -1.90984\,\lambda_2$ $\qquad$ $B_5 = -0.07639\,\lambda_2$

$B_2 = 0$ $\qquad$ $B_6 = 0$

$B_3 = -0.21220\,\lambda_2$

while $C_0 = 0$ $\qquad$ $C_4 = -0.75\lambda_2$

$C_1 = 2\lambda_1 + 3\lambda_2$ $\qquad$ $C_5 = 0.4\lambda_1 + 0.6\lambda_2$

$C_2 = -1.5\lambda_2$ $\qquad$ $C_6 = -0.5\lambda_2$

$C_3 = 0.66667\lambda_1 + \lambda_2$

In addition to these quantities required for insertion into Eq. (65) of Part I, it is also necessary to know the values of c and c' appearing therein. The values of c and c' may be obtained by use of Eqs. (66) of Part I, and the pertinent calculations result in the following values:

$$c_0 = 1 - 1.5\ A_0 - 0.5625\ A_0 = 0.88108$$

$$c_1 = 1 - 0.75A_1 - 0.28125A_1 = 0.97645$$

$$c_2 = 1 - 1.03125A_2 = 1.03277$$

$$c_3 = 1 - 1.03125A_3 = 1.00673$$

$$c_4 = 1 - 1.03125A_4 = 0.99737$$

$$c_5 = 1 - 1.03125A_5 = 0.99462$$

$$c_6 = 1 - 1.03125A_6 = 0.95723$$

while

$$c_1' = 1 - 1.03125A_7 = 0.80660$$

$$c_2' = 1 - 1.03125A_8 = 0.88652$$

$$c_3' = 1 - 1.03125A_9 = 0.88876$$

$$c_4' = 1 - 1.03125A_{10} = 0.91824$$

$$c_5' = 1 - 1.03125A_{11} = 0.94126$$

$$c_6' = 1 - 1.03125A_{12} = 0.93212$$

Finally, the coefficients $b_{r,i}$ and $b_{i,r}$, which are defined as part of Eqs. (66') in Part I, may be seen to have the values:

$b_{0,1} = 2$

$b_{0,2} = 1$

$b_{0,3} = 0.66687$

$b_{0,4} = 0.5$

$b_{0,5} = 0.4$

$b_{0,6} = 0.33333$

$b_{2,0} = 0$

$b_{2,1} = -0.66667$

$b_{2,3} = 1.2$

$b_{2,4} = 0.66667$

$b_{2,5} = 0.47619$

$b_{2,6} = 0.375$

and

$b_{4,0} = 0$

$b_{4,1} = -0.13333$

$b_{4,2} = -0.33333$

$b_{4,3} = -0.85714$

$b_{4,5} = 1.11111$

$b_{4,6} = 0.6$

$b_{6,0} = 0$

$b_{6,1} = -0.05714$

$b_{6,2} = -0.125$

$b_{6,3} = -0.22222$

$b_{6,4} = -0.4$

$b_{6,5} = -0.90909$

and likewise

$b_{1,0} = 0$

$b_{1,2} = 1.33333$

$b_{1,3} = 0.75$

$b_{1,4} = 0.53333$

$b_{1,5} = 0.41667$

$b_{1,6} = 0.34286$

$b_{3,0} = 0$

$b_{3,1} = -0.25$

$b_{3,2} = -0.8$

$b_{3,4} = 1.14286$

$b_{3,5} = 0.625$

$b_{3,6} = 0.44444$

and

$b_{5,0} = 0$

$b_{5,1} = -0.08333$

$b_{5,2} = -0.19048$

$b_{5,3} = -0.375$

$b_{5,4} = -0.88889$

$b_{5,6} = 1.09091$

In order to get a first approximation to the sought H and K coefficients, one may now make use of the above-determined quantities, and guess, as a first set of values for L_i, L_i', L_r and L_r', to be inserted into Eqs. (65) of Part I, the common value of zero. The equations for the determination of the first approximate values for the H's then turn out to be simply:

$$
\begin{aligned}
0.83108\,H_0 = {} & \\
B_0+0.47746\Big\{ & 0.46500\big[C_1+0.47746(0.08295H_0-0.07336H_2-0.01057H_4-0.00376H_6)\big] \\
& +0.08091\big[C_3+0.47746(0.02765H_0+0.13205H_2-0.06796H_4-0.01463H_6)\big] \\
& +0.02421\big[C_5+0.47746(0.01659H_0+0.05240H_2+0.08809H_4-0.05983H_6)\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
1.03277H_2 = {} & \\
B_2+0.47746\Big\{ & -0.15500\big[C_1+0.03960H_0-0.03503H_2-0.00505H_4-0.00179H_6\big] \\
& +0.14564\big[C_3+0.01320H_0+0.06305H_2-0.03245H_4-0.00698H_6\big] \\
& +0.02882\big[C_5+0.00792H_0+0.02502H_2+0.04206H_4-0.02357H_6\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
0.99737\,H_4 = {} & \\
B_4+0.47746\Big\{ & -0.03100\big[C_1+0.03960H_0-0.03503H_2-0.00505H_4-0.00179H_6\big] \\
& -0.10403\big[C_3+0.01320H_0+0.06305H_2-0.03245H_4-0.00698H_6\big] \\
& +0.06724\big[C_5+0.00792H_0+0.02502H_2+0.04206H_4-0.02857H_6\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
0.95723\,H_6 = {} & \\
B_6+0.47746\Big\{ & -0.01328\big[C_1+0.03960H_0-0.03503H_2-0.00505H_4-0.00179H_6\big] \\
& -0.02697\big[C_3+0.01320H_0+0.06305H_2-0.03245H_4-0.00698H_6\big] \\
& -0.05502\big[C_5+0.00792H_0+0.02502H_2+0.04206H_4-0.02857H_6\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
0.97645\,H_1 = {} & \\
B_1+0.47746\Big\{ & 0.16550\big[C_2+0.47746(0.25005H_1-0.08629H_3-0.01085H_5)\big] \\
& +0.04605\big[C_4+0.47746(0.10002H_1+0.12328H_3-0.05063H_5)\big] \\
& +0.02421\big[C_6+0.47746(0.06430H_1+0.04794H_3+0.06214H_5)\big]\Big\}
\end{aligned}
$$

$$1.00673\,H_3 =$$

$$B_3+0.47746\Big\{-0.09930\left[C_2+0.11939H_1-0.04120H_3-0.00518H_5\right]$$
$$+0.09868\left[C_4+0.04775H_1+0.05886H_3-0.02417H_5\right]$$
$$+0.03138\left[C_6+0.03070H_1+0.02289H_3+0.02967H_5\right]\Big\}$$

$$0.99462\,H_5 =$$

$$B_5+0.47746\Big\{-0.02364\left[C_2+0.11939H_1-0.04120H_3-0.00518H_5\right]$$
$$-0.07675\left[C_4+0.04775H_1+0.05886H_3-0.02417H_5\right]$$
$$+0.07703\left[C_6+0.03070H_1+0.02289H_3+0.02967H_5\right]\Big\}$$

These equations may be reduced in complexity, therefore, to give just the following

$$0.87169H_0+0.00505H_2+0.00188H_4+0.00100H_6=B_0+0.22202C_1+0.03863C_3+0.01156C_5$$

$$0.00190H_0+1.02546H_2+0.00131H_4+0.00074H_6=B_2-0.07401C_1+0.06954C_3+0.01376C_5$$

$$0.00100H_0+0.00181H_2+0.99434H_4+0.00054H_6=B_4-0.01480C_1-0.04967C_3+0.03210C_5$$

$$0.00063H_0+0.00125H_2+0.00065H_4+0.95638H_6=B_6-0.00634C_1-0.01288C_3-0.02627C_5$$

and likewise

$$0.96562H_1+0.00171H_3+0.00060H_5=B_1+0.07902C_2+0.02194C_4+0.01156C_6$$

$$0.00295H_1+1.00167H_3+0.00045H_5=B_3-0.04741C_2+0.04712C_4+0.01498C_6$$

$$0.00197H_1+0.00086H_3+0.99258H_5=B_5-0.01129C_2-0.03664C_4+0.03678C_6.$$

Before application of the pertinent boundary conditions, the value of λ_1 and λ_2 are unspecified parameters, so that, at this stage, the solution for H_n and K_k will be of the form $H_n = H_n^{*}\lambda_1 + H_n^{**}\lambda_2$ and $K_k=K_k^{*}\lambda_1+K_k^{**}\lambda_2$. Thus, in proceeding to the next step in finding the first approximate value for H_n one may look for it in the form $H_n^{(1)} = H_n^{*(1)}\lambda_1 + H_n^{**(1)}\lambda_2$, where the superscript "one" denotes that the values to be obtained are but the first approximations. Upon separation of the reduced equations just derived above into the part which is multiplied by λ_1 and the part which is multiplied by λ_2, therefore, it will be found that the terms linked to λ_1 give

$$0.87169H_o^{*(1)} + 0.00505H_2^{*(1)} + 0.00188H_4^{*(1)} + 0.00100H_6^{*(1)} = 3.61601$$
$$0.00190H_o^{*(1)} + 1.02546H_2^{*(1)} + 0.00131H_4^{*(1)} + 0.00074H_6^{*(1)} = -0.09616$$
$$0.00100H_o^{*(1)} + 0.00181H_2^{*(1)} + 0.99434H_4^{*(1)} + 0.00054H_6^{*(1)} = -0.04987$$
$$0.00063H_o^{*(1)} + 0.00125H_2^{*(1)} + 0.00065H_4^{*(1)} + 0.95638H_6^{*(1)} = -0.03177$$

and solution of this simultaneous set of four equations gives the $H^{*(1)}$ values as

$$H_o^{*(1)} = 4.14828$$
$$H_2^{*(1)} = -0.10146$$
$$H_4^{*(1)} = -0.05414$$
$$H_6^{*(1)} = -0.03578$$

while, because λ_1 does not appear in the expressions for B_1, B_3, B_5, C_2, C_4, and C_6, if follows that

$$H_1^{*(1)} = H_3^{*(1)} = H_5^{*(1)} = 0.$$

In similar manner it will be found that the terms multiplied by λ_2 must obey the relations:

$$0.87169H_o^{**(1)} + 0.00505H_2^{**(1)} + 0.00188H_4^{**(1)} + 0.00100H_6^{**(1)} = 5.42402$$
$$0.00190H_o^{**(1)} + 1.02546H_2^{**(1)} + 0.00131H_4^{**(1)} + 0.00074H_6^{**(1)} = -0.14423$$
$$0.00100H_o^{**(1)} + 0.00181H_2^{**(1)} + 0.99434H_4^{**(1)} + 0.00054H_6^{**(1)} = -0.07481$$
$$0.00063H_o^{**(1)} + 0.00125H_2^{**(1)} + 0.00065H_4^{**(1)} + 0.95638H_6^{**(1)} = -0.04766$$

and likewise

$$0.96562H_1^{**(1)} + 0.00171H_3^{**(1)} + 0.00060H_5^{**(1)} = -2.05064$$
$$0.00295H_1^{**(1)} + 1.00167H_3^{**(1)} + 0.00045H_5^{**(1)} = -0.18392$$
$$0.00197H_1^{**(1)} + 0.00086H_3^{**(1)} + 0.99258H_5^{**(1)} = -0.05036$$

and upon solution of these two sets of simultaneous equations, it is found that

$$H_o^{**(1)} = 6.22242$$
$$H_2^{**(1)} = -0.15218$$
$$H_4^{**(1)} = -0.08122$$
$$H_6^{**(1)} = -0.05368$$

and $$H_1^{**(1)} = -2.12365$$
$$H_3^{**(1)} = -0.17736$$
$$H_5^{**(1)} = -0.04637$$

By use of the approximation that $L_i = L_i' = L_r = L_r' = 0$ still, it is likewise seen, by use of Eqs. (65) of Part I, that the first approximate values for the K's are determined, in the present instance, by use of the following simple expressions, which have actually been employed already in the determination of the H-values:

$$K_1 = 1.23977 \left[C_1 + 0.47746(0.08295H_o - 0.07336H_2 - 0.01057H_4 - 0.00376H_6)\right]$$
$$K_3 = 1.12516 \left[C_3 + 0.47746(0.02765H_o + 0.13205H_2 - 0.06796H_4 - 0.01463H_6)\right]$$
$$K_5 = 1.06241 \left[C_5 + 0.47746(0.01659H_o + 0.05240H_2 + 0.08809H_4 - 0.05983H_6)\right]$$
$$K_2 = 1.12801 \left[C_2 + 0.47746(0.25005H_1 - 0.08629H_3 - 0.01085H_5)\right]$$
$$K_4 = 1.08904 \left[C_4 + 0.47746(0.10002H_1 + 0.12328H_3 - 0.05063H_5)\right]$$
$$K_6 = 1.07282 \left[C_6 + 0.47746(0.06430H_1 + 0.04794H_3 + 0.06214H_5)\right]$$

As was done in the case of the H's, the K's are to be separated into two parts linked to the λ_1 and λ_2 parameters, and if the superscript "one" is again used to denote the first approximation values, the solutions of the above equations are found to be:

$$K_1^{*(1)} = 2.68804$$
$$K_3^{*(1)} = 0.80677$$
$$K_5^{*(1)} = 0.45582$$
$$\text{while } K_2^{*(1)} = K_4^{*(1)} = K_6^{*(1)} = 0$$

and, likewise, the solutions for the part of the K's which is linked to the λ_2 parameter are found to be, for the first approximation:

$$K_1^{**(1)} = 4.03207$$
$$K_3^{**(1)} = 1.21015$$
$$K_5^{**(1)} = 0.68373$$

$$K_2^{**(1)} = -1.96949$$
$$K_4^{**(1)} = -0.93738$$
$$K_6^{**(1)} = -0.61219 \ .$$

Now that the first approximation values for the H's and K's are at hand, it is possible to compute the corresponding first approximation values for L and L', according to Eqs. (66') of Part I. The general equations for determining the successive L's and L''s are thus

$$L_o = -0.04567\, H_1 + 0.00145\, H_3 - 0.00042\, H_5$$
$$-1.57080(-0.37507K_1+0.11004K_2-0.07191K_3+0.03964K_4-0.02279K_5+0.02194K_6)$$
$$L_1 = -0.11532H_o+0.03531H_2-0.00039H_4-0.00251H_6$$
$$-1.57080(-0.14672K_2+0.08090K_3-0.04228K_4+0.02373K_5-0.02257K_6+0.18754K_1)$$
$$L_2 = -0.02537H_1+0.00679H_3-0.00069H_5$$
$$-1.57080(0.12502K_1-0.12944K_3+0.05286K_4-0.02713K_5+0.02468K_6+0.05502K_2)$$
$$L_3 = -0.01281H_o+0.03305H_2-0.00260H_4-0.00512H_6$$
$$-1.57080(-0.04688K_1+0.08803K_2-0.09061K_4+0.06878K_5-0.02925K_6+0.03596K_3)$$
$$L_4 = -0.00345H_1+0.00666H_3-0.00528H_5$$
$$-1.57080(0.02500K_1-0.03668K_2+0.09246K_3-0.06329K_5+0.03949K_6+0.01982K_4)$$
$$L_5 = -0.00461H_o+0.00418H_2-0.00258H_4-0.04182H_6$$
$$-1.57080(-0.01563K_1+0.02096K_2-0.04045K_3+0.07048K_4-0.07180K_6+0.01316K_5)$$
$$L_6 = -0.00138H_1+0.00081H_3-0.00526H_5$$
$$-1.57080(0.01072K_1-0.01375K_2+0.02397K_3-0.03171K_4+0.05178K_5+0.01097K_6)$$

and the similar array for the primed L's has the specific form

$$L_1' = -1.57080(-0.11532H_o-0.02118H_2+0.00163H_3+0.00034H_4-0.00043H_5+0.00237H_6$$
$$+0.02283H_1)-0.12227K_2-0.01198K_4-0.00398K_6$$
$$L_2' = +1.57080(0.05766H_o-0.03044H_1-0.00522H_3-0.00085H_4+0.00099H_5-0.00518H_6$$
$$-0.01589H_2)-0.20837K_1-0.11218K_3-0.00749K_5$$

$$L_3' = -1.57080(-0.03844H_o+0.01712H_1+0.03813H_2+0.00219H_4-0.00195H_5+0.00922H_6$$
$$-0.00217H_3)-0.11444K_2-0.08090K_4-0.00813K_6$$
$$L_4' = -1.57080(0.02883H_o-0.01218H_1-0.02118H_2+0.00746H_3+0.00463H_5-0.01659H_6$$
$$+0.00064H_4)-0.02834K_1-0.11007K_3-0.05767K_5$$
$$L_5' = -1.57080(-0.02306H_o+0.00951H_1+0.01513H_2-0.00408H_3-0.00284H_4+0.03770H_6$$
$$+0.00104H_5)-0.01447K_2-0.08026K_4-0.06636K_6$$
$$L_6' = -1.57080(0.01922H_o-0.00783H_1-0.01192H_2+0.00290H_3+0.00153H_4-0.00569H_5$$
$$+0.00691H_6)-0.01133K_1-0.01332K_3-0.05743K_5.$$

Treating these L and L' values in the same way that the H's and K's were handled, by splitting them up into two parts, the evaluations will be sought in the form

$$L = L^{*}\lambda_1 + L^{**}\lambda_2 \qquad \text{and} \qquad L' = L'^{*}\lambda_1 + L'^{**}\lambda_2.$$

The first approximations may now be obtained directly, therefore, by substituting the above-determined first approximation values for H^*, H^{**}, K^*, and K^{**} into the equations for L and L' just set down above. The numerical results obtained when this procedure is carried through are found to be

$$L_o^{*} = 1.69114 \qquad L_o^{**} = 3.05334$$
$$L_1^{*} = -1.39323 \qquad L_1^{**} = -2.62770$$
$$L_2^{*} = -0.34441 \qquad L_2^{**} = -0.19215$$
$$L_3^{*} = 0.04695 \qquad L_3^{**} = 0.18123$$
$$L_4^{*} = -0.17742 \qquad L_4^{**} = -0.30606$$
$$L_5^{*} = 0.08992 \qquad L_5^{**} = 0.23447$$
$$L_6^{*} = -0.11272 \qquad L_6^{**} = -0.24472$$

and likewise it is found that

$$L_1'^{*} = 0.74822 \qquad L_1'^{**} = 1.45339$$
$$L_2'^{*} = -1.03263 \qquad L_2'^{**} = -1.65188$$
$$L_3'^{*} = 0.25727 \qquad L_3'^{**} = 0.74846$$
$$L_4'^{*} = -0.38338 \qquad L_4'^{**} = -0.61327$$
$$L_5'^{*} = 0.15455 \qquad L_5'^{**} = 0.40684$$
$$L_6'^{*} = -0.19400 \qquad L_6'^{**} = -0.31672$$

If the method of successive approximations is now carried one step farther by use of the just found values for L and L', it will be seen that the two sets of simultaneous equations, from which the second approximation values for H and K may be determined, become now

$$0.87169H_0^{*(2)} +0.00505H_2^{*(2)} +0.00188H_4^{*(2)} +0.00100H_6^{*(2)} = 3.82903$$
$$0.00190H_0^{*(2)} +1.02546H_2^{*(2)} +0.00131H_4^{*(2)} +0.00074H_6^{*(2)} = -0.13944$$
$$0.00100H_0^{*(2)} +0.00181H_2^{*(2)} +0.99434H_4^{*(2)} +0.00054H_6^{*(2)} = -0.07224$$
$$0.00063H_0^{*(2)} +0.00125H_2^{*(2)} +0.00065H_4^{*(2)} +0.95638H_6^{*(2)} = -0.04600$$

and for the set with odd subscripts one has

$$0.96562H_1^{*(2)} +0.00171H_3^{*(2)} +0.00060H_5^{*(2)} = -0.16932$$
$$0.00295H_1^{*(2)} +1.00167H_3^{*(2)} +0.00045H_5^{*(2)} = 0.00354$$
$$0.00197H_1^{*(2)} +0.00086H_3^{*(2)} +0.99258H_5^{*(2)} = 0.01237 .$$

Upon solving these sets of equations it is found that the second approximation values for the H's, at least that part of the H which is related to the λ_1 parameter, are as follows:

$$H_0^{*(2)} = 4.39371 \qquad H_1^{*(2)} = -0.17537$$
$$H_2^{*(2)} = -0.14398 \qquad H_3^{*(2)} = 0.00904$$
$$H_4^{*(2)} = -0.07678 \qquad H_5^{*(2)} = 0.01280 .$$
$$H_6^{*(2)} = -0.05075$$

By following the same procedure as utilized in the previous approximations, the corresponding values of K may be obtained from use of these H values just set

down above, and upon carrying out the pertinent calculations it is found that the second approximations for the K's, at least the part linked to the λ_1 parameter, turn out to be

$$K_1^{*(2)} = 2.80785 \qquad K_4^{*(2)} = -0.05647$$
$$K_2^{*(2)} = -0.15688 \qquad K_5^{*(2)} = 0.47491$$
$$K_3^{*(2)} = 0.84134 \qquad K_6^{*(2)} = -0.02887 \; .$$

In an analogous fashion the H values which are linked to the λ_2 parameter are determined by solution of the array of equations

$$0.87169H_0^{**(2)}+0.00505H_2^{**(2)}+0.00138H_4^{**(2)}+0.00100H_6^{**(2)}= 5.81265$$
$$0.00190H_0^{**(2)}+1.02546H_2^{**(2)}+0.00131H_4^{**(2)}+0.00074H_6^{**(2)}= -0.17182$$
$$0.00100H_0^{**(2)}+0.00131H_2^{**(2)}+0.99434H_4^{**(2)}+0.00054H_6^{**(2)}= -0.11489$$
$$0.00063H_0^{**(2)}+0.00125H_2^{**(2)}+0.00065H_4^{**(2)}+0.95638H_6^{**(2)}= -0.07892$$

and

$$0.96562H_1^{**(2)}+0.00171H_3^{**(2)}+0.00060H_5^{**(2)} = -2.36698$$
$$0.00295H1^{**(2)}+1.00167H_3^{**(2)}+0.00045H_5^{**(2)} = -0.15817$$
$$0.00197H_1^{**(2)}+0.00086H_3^{**(2)}+0.99258H_5^{**(2)} = -0.02028.$$

The solution of these equations will be found to give the following values for the $H^{**(2)}$ values:

$$H_0^{**(2)} = 6.66966 \qquad H_1^{**(2)} = -2.45098$$
$$H_2^{**(2)} = -0.17969 \qquad H_3^{**(2)} = -0.15068$$
$$H_4^{**(2)} = -0.12188 \qquad H_5^{**(2)} = -0.01544 \; .$$
$$H_6^{**(2)} = -0.08660$$

By reuse of these values in the way that it was done for the first approximation, it will be found that the corresponding evaluations for the $K^{**(2)}$ values are

$$K_1^{**(2)} = 4.26093$$
$$K_2^{**(2)} = -2.22738$$
$$K_3^{**(2)} = 1.31258$$
$$K_4^{**(2)} = -1.02963$$
$$K_5^{**(2)} = 0.73521$$
$$K_6^{**(2)} = -0.66007$$

There appears to be little reason to proceed to any further degree of approximation by continuing the above process, so that consequently it may now be taken for granted that the H and K values pertinent to the case in hand, have the following evaluations:

$$H_o = 4.39371\ \lambda_1 + 6.66966\ \lambda_1$$
$$H_1 = -0.17537\ \lambda_1 - 2.45098\ \lambda_2$$
$$H_2 = -0.14398\ \lambda_1 - 0.17969\ \lambda_2$$
$$H_3 = 0.00904\ \lambda_1 - 0.15068\ \lambda_2$$
$$H_4 = -0.07678\ \lambda_1 - 0.12188\ \lambda_2$$
$$H_5 = 0.01280\ \lambda_1 - 0.01544\ \lambda_2$$
$$H_6 = -0.05075\ \lambda_1 - 0.08660\ \lambda_2$$

while

$$K_1 = 2.80785\ \lambda_1 + 4.26093\ \lambda_2$$
$$K_2 = -0.15688\ \lambda_1 - 2.22738\ \lambda_2$$
$$K_3 = 0.84134\ \lambda_1 + 1.31258\ \lambda_2$$
$$K_4 = -0.05647\ \lambda_1 - 1.02963\ \lambda_2$$
$$K_5 = 0.47491\ \lambda_1 + 0.73521\ \lambda_2$$
$$K_6 = -0.02887\ \lambda_1 - 0.66007\ \lambda_2$$

The functional relationship which describes how the slope of the external contour of the duct must behave under the thus-far imposed general conditions may now be written down by aid of the A, H, and K values that have been

determined above. If the sought η-function is broken up into the part linked to the λ_1-parameter and the part linked to the λ_2-parameter by use of the definition that

$$\eta(z) = \eta_1(z)\cdot\lambda_1 + \eta_2(z)\cdot\lambda_2$$

then it follows, from use of Eqs. (56) and the one preceding it in Part I, that the respective parts of the η-function have the form:

$$\begin{aligned}\eta_1(z) = \frac{1}{1+0.25z} &+ 0.12096-0.00191\cos\theta + 0.00218\cos 2\theta \\ &-0.00003\cos 3\theta -0.00010\cos 4\theta +0.00003\cos 5\theta -0.00100\cos 6\theta \\ &+0.25142\sin\theta -0.00824\sin 2\theta +0.04333\sin 3\theta -0.00214\sin 4\theta \\ &+0.01292\sin 5\theta -0.00091\sin 6\theta\end{aligned}$$

and

$$\begin{aligned}\eta_2(z) = \frac{z}{1+0.25z} &+ 0.18362 -0.02672\cos\theta + 0.00273\cos 2\theta +0.00047\cos 3\theta \\ &-0.00015\cos 4\theta -0.00004\cos 5\theta -0.00171\cos 6\theta +0.38153\sin\theta \\ &-0.11703\sin 2\theta +0.06760\sin 3\theta -0.03898\sin 4\theta +0.02000\sin 5\theta \\ &-0.02074\sin 6\theta.\end{aligned}$$

7. Determination of the λ_1 and λ_2 Parameters

The stage has now been reached in the development of the computations where it is possible to apply the final specific perimetral conditions which were selected for study in the Introduction. These cases will now be restated, and the final step in the determination of the duct's slope will then be carried out.

Case (a): In this case the sole condition to be imposed is that the duct should close, i.e., it is merely assumed that the "end-points" N_1 and N_2 at the lip and at the tail end of the sought duct's meridional contour are

to belong to the frustum of a cone section which constitutes the inner duct whose large end and small end have radii in the ratio $\frac{R_2}{R_1} = 1 + \beta \cdot \frac{L}{R_1}$, where R_1 is here equal to the radial coordinate of the N_1 lip-point and R_2 is equal to the radial coordinate of the N_2 tail-point of the sought external contour.

In this instance, therefore, the appropriate conditions on the constants λ_1 and λ_2 are that λ_2 is to be taken equal to zero, while λ_1 is found by making use of Eq. (68) of Part I, which, in the present notation, becomes

$$\frac{1}{B} \cdot \frac{R_2 - R_1}{R_1} = \frac{1}{B} \frac{\beta L}{R_1} = \beta \ell_1 = \lambda_1 \int_0^{\ell_1} \eta_1(\zeta)\, d\gamma \ .$$

Upon use of the appropriate concrete data applying to the case in hand, it is found that

$$\int_0^3 \eta_1(\zeta)\, d\gamma = 3.11405$$

and therefore the value of λ_1 turns out to be $\lambda_1 = \frac{3\beta}{3.11405}$.

In the Normalizing Mach Number Case, therefore, where one has that $B = 1$ and that $\beta = 0.25$, the corresponding value of λ_1 is $\lambda_1 = 0.24084$. The slope function which applies in this specific case then becomes

$$\eta(z) = \frac{0.24084}{1+0.25z} + 0.02913 - 0.00046 \cos\theta + 0.0052 \cos 2\theta - 0.00001 \cos 3\theta$$
$$- 0.00002 \cos 4\theta + 0.00001 \cos 5\theta - 0.00024 \cos 6\theta + 0.06055 \sin\theta$$
$$- 0.00198 \sin 2\theta + 0.01044 \sin 3\theta - 0.00051 \sin 4\theta + 0.00311 \sin 5\theta$$
$$- 0.00022 \sin 6\theta \ .$$

The actual R coordinates of the sought duct are then given by the integral, which takes the form

$$\int_0^{\gamma} \eta(s)ds = \frac{R-R_1}{BR_1} = 0.96336 \log_e (1+0.25\gamma) +0.02913\gamma -0.00044 \sin\left(\frac{\pi\gamma}{3}\right)$$
$$+ 0.00025 \sin \left(\frac{2\pi\gamma}{3}\right) -0.00000 \sin (\pi\gamma) -0.00000 \sin \left(\frac{4\pi\gamma}{3}\right)$$
$$+ 0.00000 \sin \left(\frac{5\pi\gamma}{3}\right) -0.00004 \sin (2\pi\gamma)$$
$$+ 0.05782 \left[1-\cos \left(\frac{\pi\gamma}{3}\right)\right] -0.00094 \left[1-\cos \left(\frac{2\pi\gamma}{3}\right)\right]$$
$$+ 0.00332 \left[1-\cos (\pi\gamma)\right] -0.00012 \left[1-\cos \left(\frac{4\pi\gamma}{3}\right)\right]$$
$$+ 0.00059 \left[1-\cos \left(\frac{5\pi\gamma}{3}\right)\right] -0.00003 \left[1-\cos (2\pi\gamma)\right] . \qquad (11)$$

It will be instructive to find out how the two-dimensional solution compares with this result. In other words a computation will now be made, using the same perimetral condition as employed in the above example, but this time the simple Ackeret formula will be utilized for relating the local pressures to the local slopes of the sought duct's contour. Under these circumstances the local slopes of the contour are described by the formula

$$\eta = \frac{\lambda_1}{1+0.25z}$$

and consequently the value of λ_1 is found to be $\lambda_1 = 0.33505$, because here it is true that

$$\int_0^3 \eta_1(\gamma)d\gamma = 2.23348 .$$

The formula for the "two-dimensional" duct coordinates thus becomes

$$\frac{R-R_1}{BR_1} = \int_0^{\gamma} \eta(s)ds = 1.34020 \log_e (1+0.25\gamma) . \qquad (12)$$

The radial coordinates, $\frac{R-R_1}{BR_1}$, which apply to the duct in question, which has minimum drag, with no perimetral condition imposed other than the one of closure, and in the Normalizing Mach No. Case, have been computed for certain selected abscissa stations, γ , along the duct's length. The results for both the three-dimensional and the two-dimensional cases are listed in Table XI below, while the plot of the meridional line now obtained is shown in Fig. 5 for both cases also.

TABLE XI

DUCT COORDINATES FOR CASE OF NO PERIMETRAL CONDITION, SAVE THAT OF CLOSURE - NORMALIZING MACH NO.

γ	$\frac{R-R_1}{BR_1}$ By Use of Formula (11)	$\frac{R-R_1}{BR_1}$ By Use of Formula (12)
0	0	0
0.4	0.11113	0.12773
0.8	0.22363	0.24434
1.2	0.33155	0.35161
1.6	0.43540	0.45094
1.8	0.48555	0.49796
2.2	0.58181	0.58734
2.4	0.62814	0.62989
2.8	0.71372	0.71115
3.0	0.75000	0.75000

In the Small Conical Diffusor Angle Case the value of λ_1 will be changed from what it was above, but the same procedure will otherwise be valid. In this new case, therefore, one has that $B = 1.66667$ and $\beta = 0.15$; so that one may now proceed as in the former analogous case to obtain

$$\lambda_1 = \frac{0.45}{3.11405} = 0.14451$$

DUCT CONTOUR FOR LEAST DRAG IN CASE OF NO PERIMETRAL CONDITION SAVE THAT OF CLOSURE - NORMALIZING MACH NO. CASE

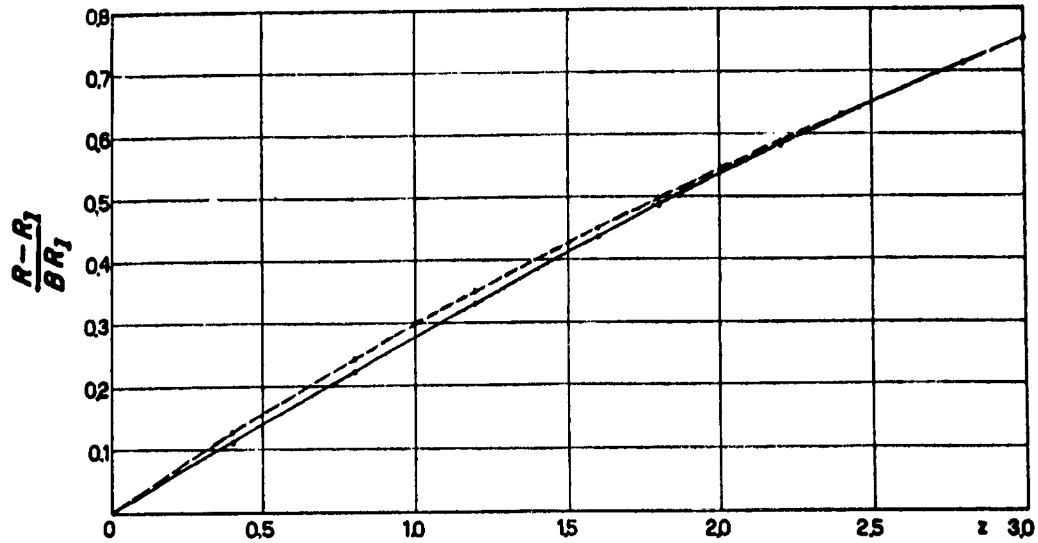

FIGURE 5

DUCT CONTOUR FOR LEAST DRAG IN CASE OF NO PERIMETRAL CONDITION SAVE THAT OF CLOSURE - SMALL CONICAL DIFFUSOR ANGLE CASE

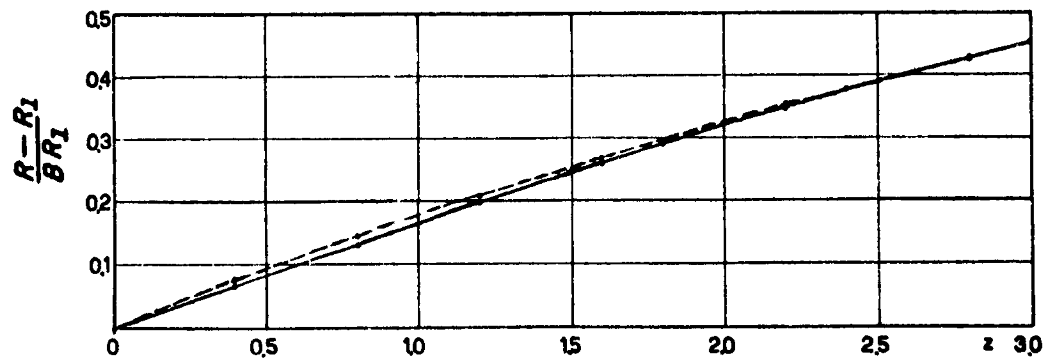

FIGURE 6

LEGEND:

—— MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT GEOMETRY BY MEANS OF EQ. (47) OF PART I

--- MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT PLANAR ELEMENTS THROUGH ACKERET'S TWO-DIMENSIONAL FORMULA

and thus the corresponding description of how the duct contour behaves is provided by the function

$$\frac{R-R_1}{BR_1} = 0.57804 \log_e (1+0.25\gamma) + 0.01748\gamma - 0.00027 \sin\left(\frac{\pi\gamma}{3}\right)$$
$$+ 0.00015 \sin\left(\frac{2\pi\gamma}{3}\right) - 0.00000 \sin(\pi\gamma) - 0.00000 \sin\left(\frac{4\pi\gamma}{3}\right)$$
$$+ 0.00000 \sin\left(\frac{5\pi\gamma}{3}\right) - 0.00002 \sin(2\pi\gamma)$$
$$+ 0.03469 \left[1-\cos\left(\frac{\pi\gamma}{3}\right)\right] - 0.00057 \left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right]$$
$$+ 0.00199 \left[1-\cos(\pi\gamma)\right] - 0.00007 \left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right]$$
$$+ 0.00036 \left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right] - 0.00002 \left[1-\cos(2\pi\gamma)\right] \quad . \qquad (11')$$

By use of the Ackeret formula under the same conditions one would obtain on the other hand that

$$\lambda_1 = \frac{0.45}{2.23848} = 0.20103$$

and thus the ordinates in this case would be given by the expression

$$\frac{R-R_1}{BR_1} = 0.80412 \log_e (1+0.25z). \qquad (12')$$

The radial coordinates, $\frac{R-R_1}{BR_1}$, which apply to the duct in question, which has minimum drag, with no perimetral condition imposed other than the one of closure, in the Small Conical Diffusor Angle Case, have been computed for certain selected abscissa stations, z, along the duct's length. The results for both the three-dimensional and the two-dimensional cases are listed in Table XII below, while the plots of the meridional line obtained by use of these data are shown in Fig. 6 for both instances.

TABLE XII

DUCT COORDINATES FOR CASE OF NO PERIMETRAL CONDITION, SAVE THAT OF CLOSURE - SMALL CONICAL DIFFUSOR ANGLE

z	$\frac{R - R_1}{BR_1}$	$\frac{R - R_1}{BR_1}$
	By Use of Formula (11')	By Use of Formula (12')
0	0	0
0.4	0.06669	0.07664
0.8	0.13418	0.14661
1.2	0.19892	0.21097
1.6	0.26123	0.27056
1.8	0.29133	0.29878
2.2	0.34909	0.35241
2.4	0.37689	0.37794
2.8	0.42825	0.42669
3.0	0.45000	0.45000

Case (b): "No Increase in Area Over that Occupied by Frustum-of-Cone Basic Shape". In addition to the closure condition employed in Case (a), it is now further postulated that the area enclosed between the sought contour and the x-axis is to be equal to the area which would be enclosed between this x-axis and the straight line segment joining the points N_1 and N_2.

Under this general perimetral condition, the λ_1 and λ_2 parameters will have to be such as to permit Eqs. (68) and (69) of Part I to be satisfied. In the present instance, therefore, it will be required that the specific form of Eqs. (68) to be satisfied should be

$$\lambda_1 \int_0^3 \eta_1(\zeta)\, d\zeta + \lambda_2 \int_0^3 \eta_2(\zeta)\, d\zeta = \beta \ell_1$$

while, because the constant C appearing in Eq. (69) takes on the specific value $C = \frac{\beta}{2} \cdot \frac{L^2}{B^2 R_1^2}$ when the area enclosed by the sought contour is not increased over the area enclosed by the straight line chord between the end-points N_1 and N_2 and the x-axis, it follows that in this present instance Eq. (69) reduces to just

$$\lambda_1 \int_0^{\ell_1} d\zeta \int_0^{\zeta} \eta_1(s)\, ds + \lambda_2 \int_0^{\ell_1} d\zeta \int_0^{\zeta} \eta_2(s)\, ds = \frac{\beta}{2} \ell_1^2 .$$

In the Normalizing Mach No. Case for which B = 1, and $\beta = 0.25$, therefore, the evaluated forms of the above perimetral conditions become

$$3.11405\,\lambda_1 + 4.37629\,\lambda_2 = 0.75$$

$$4.96539\,\lambda_1 + 5.06409\,\lambda_2 = 1.125$$

from which it follows that the pertinent values of λ_1 and λ_2 are

$$\lambda_1 = 0.18879 \quad \text{and} \quad \lambda_2 = 0.03704 .$$

Likewise, in the Small Conical Diffusor Angle Case, for which B = 1.66667 and $\beta = 0.15$, it is readily seen that the evaluated forms of the above-stated perimetral conditions are

$$3.11405\ \lambda_1 + 4.37629\ \lambda_2 = 0.45$$

$$4.96539\ \lambda_1 + 5.06409\ \lambda_2 = 0.675$$

from which it follows that

$$\lambda_1 = 0.11328 \text{ and } \quad \lambda_2 = 0.02222.$$

Upon insertion of the λ-values into the expression for the sought slope function it is then found that

In the Normalizing Mach No. Case:

$$\begin{aligned}\eta(\zeta) = \frac{0.18879}{1+0.25\zeta} + \frac{0.03704\zeta}{1+0.25\zeta} &+0.02964 - 0.00135\cos\left(\frac{\pi\zeta}{3}\right)\\ &+0.00051\cos\left(\frac{2\pi\zeta}{3}\right)+0.00001\cos(\pi\zeta)-0.00004\cos\left(\frac{4\pi\zeta}{3}\right)\\ &+0.00000\cos\left(\frac{5\pi\zeta}{3}\right)-0.00025\cos(2\pi\zeta)+0.06160\sin\left(\frac{\pi\zeta}{3}\right)\\ &-0.00589\sin\left(\frac{2\pi\zeta}{3}\right)+0.01068\sin(\pi\zeta)-0.00185\sin\left(\frac{4\pi\zeta}{3}\right)\\ &+0.00318\sin\left(\frac{5\pi\zeta}{3}\right)-0.00094\sin(2\pi\zeta)\ .\end{aligned}$$

The actual R coordinates of the sought duct may then be obtained through integration, with the result that

$$\begin{aligned}\frac{R-R_1}{BR_1} = 0.16252\log_e(1+0.25\zeta) &+0.17780\zeta - 0.00129\sin\left(\frac{\pi\zeta}{3}\right)\\ +0.00024\sin\left(\frac{2\pi\zeta}{3}\right)&-0.00000\sin(\pi\zeta) - 0.00001\sin\left(\frac{4\pi\zeta}{3}\right)\\ +0.00000\sin\left(\frac{5\pi\zeta}{3}\right)&-0.00004\sin(2\pi\zeta) + 0.05882\left[1-\cos\left(\frac{\pi\zeta}{3}\right)\right]\\ -0.00281\left[1-\cos\left(\frac{2\pi\zeta}{3}\right)\right]&+0.00340\left[1-\cos(\pi\zeta)\right] - 0.00044\left[1-\cos\left(\frac{4\pi\zeta}{3}\right)\right]\\ +0.00061\left[1-\cos\left(\frac{5\pi\zeta}{3}\right)\right]&-0.00015\left[1-\cos(2\pi\zeta)\right]\end{aligned} \tag{13}$$

In the Small Conical Diffusor Angle Case:

The analógous formula for the slope-function is

$$\begin{aligned}\eta(z) = \frac{0.11328}{1+0.25z} + \frac{0.02222z}{1+0.25z} &+0.01778-0.00081\cos\theta + 0.00031\cos 2\theta\\ &+0.00001\cos 3\theta - 0.00001\cos 4\theta + 0.000000\cos 5\theta\\ &-0.00015\cos 6\theta + 0.03696\sin\theta - 0.00353\sin 2\theta\\ &+0.00641\sin 3\theta - 0.00111\sin 4\theta + 0.00191\sin 5\theta - 0.00056\sin 6\theta\end{aligned}$$

while the actual duct coordinates are again obtained by integration of the above equation, which gives the result that

$$\frac{R-R_1}{BR_1} = 0.09760 \log_e (1+0.25z) + 0.10666z - 0.00077 \sin\left(\frac{\pi z}{3}\right)$$
$$+ 0.00015 \sin\left(\frac{2\pi z}{3}\right) - 0.00000 \sin(\pi z) - 0.00000 \sin\left(\frac{4\pi z}{3}\right)$$
$$+ 0.00000 \sin\left(\frac{5\pi z}{3}\right) - 0.00002 \sin(2\pi z) + 0.03529 \left[1-\cos\frac{\pi z}{3}\right]$$
$$- 0.00168 \left[1-\cos\left(\frac{2\pi z}{3}\right)\right] + 0.00204 \left[1-\cos(\pi z)\right] - 0.00026 \left[1-\cos\left(\frac{4\pi z}{3}\right)\right]$$
$$+ 0.00036 \left[1-\cos\left(\frac{5\pi z}{3}\right)\right] - 0.00009 \left[1-\cos(2\pi z)\right] \tag{14}$$

If, on the other hand, the formula for the two-dimensional solution based on use of Ackeret's formula is to be sought, one has directly that

$$2.23848\ \lambda_1 + 3.04608\ \lambda_2 = 0.75$$
$$3.66928\ \lambda_1 + 3.32288\ \lambda_2 = 1.125$$

provided it is assumed that $B = 1$, and $\beta = 0.25$, and thus it develops that

<u>In the Normalizing Mach No. Case</u>:

The specific λ-values turn out to be

$$\lambda_1 = 0.25 \qquad \text{and} \qquad \lambda_2 = 0.0625$$

and thus the pertinent slope function is just

$$\eta(z) = \frac{0.25}{1+0.25z} + 0.0625 \frac{z}{1+0.25z} \tag{13'}$$

<u>In the Small Conical Diffusor Angle Case</u>

The equations defining the λ's take the evaluated form

$$2.23848\ \lambda_1 + 3.04608\ \lambda_2 = 0.45$$
$$3.66728\ \lambda_1 + 3.32288\ \lambda_2 = 0.675$$

provided it is assumed that $B = 1.66667$ and $\beta = 0.15$, and thus it is found that the specific λ-values applicable to this case are

$$\lambda_1 = 0.15 \qquad \text{and} \qquad \lambda_2 = 0.0375$$

and, consequently, the slope function applying to the case is

$$\eta(z) = \frac{0.15}{1+0.25z} + 0.0375 \frac{z}{1+0.25z} \ . \tag{14'}$$

The actual radial coordinates for the duct giving minimum drag in this case of no increase in area over that occupied by the frustum-of-a-cone basic shape, and under the specific conditions of either a normalizing Mach No. or a small conical diffusor angle, have been calculated by means of Eqs. (13) and (14) and with the integrals of Eqs.(13') and (14'), and the final values obtained are listed here in Table XIII, while the results have also been plotted in Figs.7 and 8.

TABLE XIII

DUCT COORDINATES FOR CASE OF NO INCREASE IN AREA OVER THAT OCCUPIED BY THE FRUSTUM-OF-A-CONE BASIC SHAPE

γ	Normalizing Mach No. Case B = 1.0 and β = 0.25		Small Conical Diffusor Angle Case B = 1.66667 and β = 0.15	
	$\frac{R-R_1}{BR_1}$		$\frac{R-R_1}{BR_1}$	
	By Use of Formula (13)	By Use of Formula (13')	By Use of Formula (14)	By Use of Formula (14')
0	0	0	0	0
0.4	0.09289	0.1	0.05575	0.06
0.8	0.19364	0.2	0.11619	0.12
1.2	0.29619	0.3	0.17773	0.18
1.6	0.40021	0.4	0.24013	0.24
1.8	0.45246	0.45	0.27147	0.27
2.2	0.55620	0.55	0.33373	0.33
2.4	0.60785	0.6	0.36470	0.36
2.8	0.70669	0.7	0.42398	0.42
3.0	0.75000	0.75	0.45000	0.45

DUCT CONTOUR FOR LEAST DRAG IN CASE OF NO INCREASE IN AREA OVER THAT OCCUPIED BY THE FRUSTUM-OF-A-CONE BASIC SHAPE

NORMALIZING MACH NO. CASE

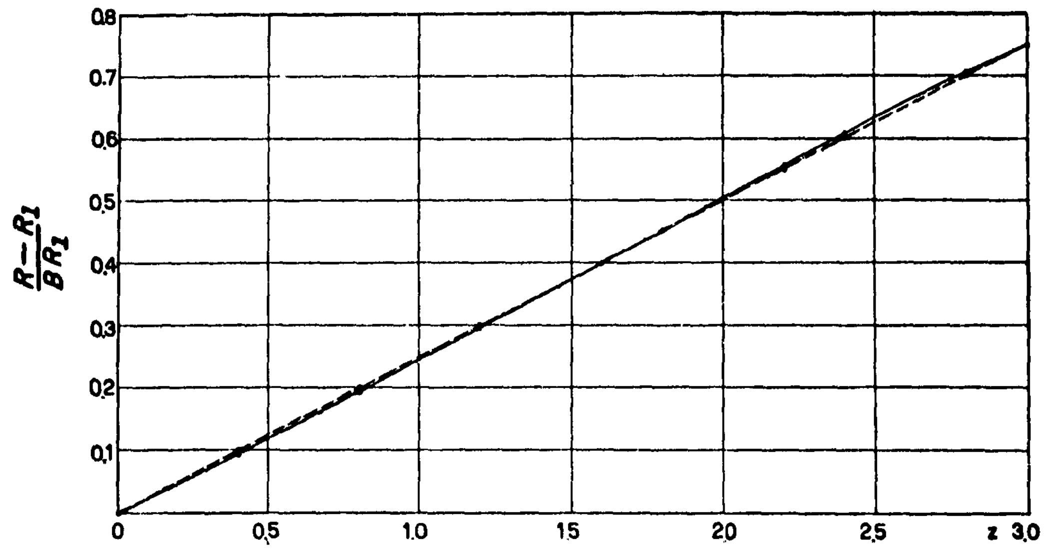

FIGURE 7

DUCT CONTOUR FOR LEAST DRAG IN CASE OF NO INCREASE IN AREA OVER THAT OCCUPIED BY THE FRUSTUM-OF-A-CONE BASIC SHAPE — SMALL CONICAL DIFFUSOR ANGLE CASE

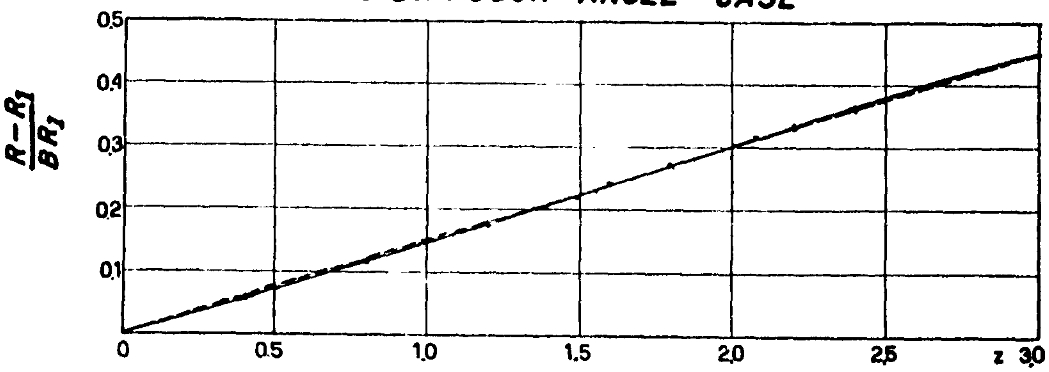

FIGURE 8

LEGEND:

—— MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT GEOMETRY BY MEANS OF EQ. (47) OF PART I

--- MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT PLANAR ELEMENTS THROUGH ACKERET'S TWO-DIMENSIONAL FORMULA

In the small conical diffusor angle case, on the other hand, the sought values of λ_1 and λ_2 are determined by the relations

$$3.11405\ \lambda_1 + 4.37629\ \lambda_2 = 0.45$$
$$4.96539\ \lambda_1 + 5.06409\ \lambda_2 = 0.9756$$

so that when the solution of this pair of equations is obtained it is seen that

$$\lambda_1 = 0.33399 \text{ and } \lambda_2 = 0.13483$$

in this instance.

It follows, consequently, that the desired slope function is

$$\begin{aligned}\eta(z) = {} & \frac{0.33399}{1+0.25z} - 0.13483\frac{z}{1+0.25z} + 0.01564 \\ & +0.00296\cos\theta + 0.00036\cos 2\theta - 0.00007\cos 3\theta \\ & -0.00001\cos 4\theta + 0.00001\cos 5\theta - 0.00010\cos 6\theta \\ & +0.03253\sin\theta + 0.01303\sin 2\theta + 0.00536\sin 3\theta \\ & +0.00454\sin 4\theta + 0.00162\sin 5\theta + 0.00249\sin 6\theta.\end{aligned}$$

The equations defining the actual external contour of the duct giving minimum drag under the above-stated conditions then may be written down, by integration of the slope functions just derived. When this is done it turns out that

<u>In the Normalizing Mach No. Case</u>:

$$\begin{aligned}\frac{R-R_1}{BR_1} = {} & 3.55836\log_e(1+0.25\gamma) - 0.45258\gamma \\ & +0.00231\sin\left(\frac{\pi\gamma}{3}\right) + 0.00027\left(\frac{2\pi\gamma}{3}\right) - 0.00002\sin(\pi\gamma) \\ & -0.00000\sin\left(\frac{4\pi\gamma}{3}\right) + 0.00000\sin\left(\frac{5\pi\gamma}{3}\right) - 0.00003\sin(2\pi\gamma) \\ & +0.05459\left[1-\cos\left(\frac{\pi\gamma}{3}\right)\right] + 0.00509\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right] + 0.00306\left[1-\cos(\pi\gamma)\right] \\ & +0.00091\left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right] + 0.00055\left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right] + 0.00034\left[1-\cos(2\pi\gamma)\right]\end{aligned} \tag{15}$$

while

In the Small Conical Diffusor Angle Case:

$$\frac{R-R_1}{BR_1} = 3.49324 \log_e (1+0.25\gamma) -0.52368\gamma$$
$$+0.00283 \sin\left(\frac{\pi\gamma}{3}\right) +0.00017 \sin\left(\frac{2\pi\gamma}{3}\right) -0.00002 \sin(\pi\gamma)$$
$$+0.00000 \sin\left(\frac{4\pi\gamma}{3}\right) -0.00000 \sin\left(\frac{5\pi\gamma}{3}\right) -0.00002 \sin(2\pi\gamma)$$
$$+0.03106\left[1-\cos\left(\frac{\pi\gamma}{3}\right)\right]+0.00622\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right]+0.00171\left[1-\cos(\pi\gamma)\right]$$
$$+0.00108\left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right]+0.00031\left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right]+0.00040\left[1-\cos(2\pi\gamma)\right] \quad (16)$$

On the other hand, if the formula for the two-dimensional solution, that is based on use of Ackeret's formula for relating the local pressures to the local duct slopes, is employed, then one finds that

In the Normalizing Mach No. Case:

$$2.23848\ \lambda_1 +3.04608\ \lambda_2 = 0.75$$
$$3.66928\ \lambda_1 +3.32288\ \lambda_2 = 1.4256$$

from whence it follows that

$$\lambda_1 = 0.49489 \text{ and } \lambda_2 = -0.11746$$

in this instance.

The related slope function for this example then takes the form

$$\eta(z) = \frac{0.49489}{1+0.25z} -0.11746\frac{z}{1+0.25z}$$

and the corresponding actual radial coordinates are found by integrating the slope function, to wit

$$\frac{R-R_1}{BR_1} = 3.85892 \log_e (1+0.25z) -0.46984\ z. \quad (15')$$

In the Small Conical Diffusor Angle Case:

$$2.23848\ \lambda_1 + 3.04608\ \lambda_2 = 0.45$$
$$3.66928\ \lambda_1 + 3.32288\ \lambda_2 = 0.9756$$

and consequently the λ values valid in these circumstances are

$$\lambda_1 = 0.39490 \text{ and } \lambda_2 = -0.14247.$$

The sought slope function may then be written down straight off as

$$\eta(z) = \frac{0.39490}{1+0.25z} - 0.14747 \frac{z}{1+0.25z}$$

so that the corresponding definition of the actual radial coordinates comes out to be

$$\frac{R-R_1}{BR_1} = 3.85912 \log_e (1+0.25z) - 0.56988z. \qquad (16')$$

The radial coordinates which apply to the duct in question, which has minimum drag in the case where the area enclosed between the sought duct contour and the chord line joining the N_1 and N_2 end-points is equal to the area enclosed between this N_1 and N_2 chord line and the parabolic curve defined by the equation

$$\frac{R}{R_1} = 1+ \beta \frac{x}{R_1} + 0.2 \frac{x}{R_1} - 0.2 \frac{x^2}{R_1^2} \cdot \frac{R_1}{L}$$

have been calculated by use of Eqs. (15), (16), (15'), and (16') under the specific conditions that a normalizing Mach No. is being used or that a small conical diffusor angle is being treated, and the results are entered into Table XIV below. The results have also been illustrated by means of the plots of Figs. 9 and 10.

TABLE XIV

DUCT COORDINATES FOR CASE WHERE SOUGHT BEST SHAPE ENCLOSES SAME AREA AS THAT COMPRISED BETWEEN N_1N_2 CHORDLINE AND AN ARBITRARILY CHOSEN PARABOLIC CONTOUR

γ	Normalizing Mach No. Case B=1.0 and β = 0.25		Small Conical Diffusor Angle Case B = 1.66667 and β = 0.15	
	$\frac{R-R_1}{BR_1}$		$\frac{R-R_1}{BR_1}$	
	By Use of Formula (15)	By Use of Formula (15')	By Use of Formula (16)	By Use of Formula (16')
0	0	0	0	0
0.4	0.17018	0.17986	0.13302	0.13986
0.8	0.32078	0.32769	0.24334	0.24769
1.2	0.44614	0.44862	0.32768	0.32862
1.6	0.54943	0.54667	0.38937	0.38667
1.8	0.59279	0.58811	0.41183	0.40811
2.2	0.66482	0.65752	0.44233	0.43752
2.4	0.69395	0.68608	0.45081	0.44607
2.8	0.73666	0.73211	0.45401	0.45210
3.0	0.75000	0.75000	0.45000	0.45000

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST DUCT SHAPE ENCLOSES THE SAME AREA AS THAT COMPRISED BETWEEN THE $N_1 N_2$ CHORDLINE AND THE PARABOLIC CURVE DEFINED ACCORDING TO THE EQUATION

$$\frac{R}{R_1} = 1 + \beta\frac{x}{R_1} + 0.2\,\frac{x}{R_1} - 0.2\,\frac{x^2}{R_1^2}\cdot\frac{R_1}{L}$$

NORMALIZING MACH NO. CASE

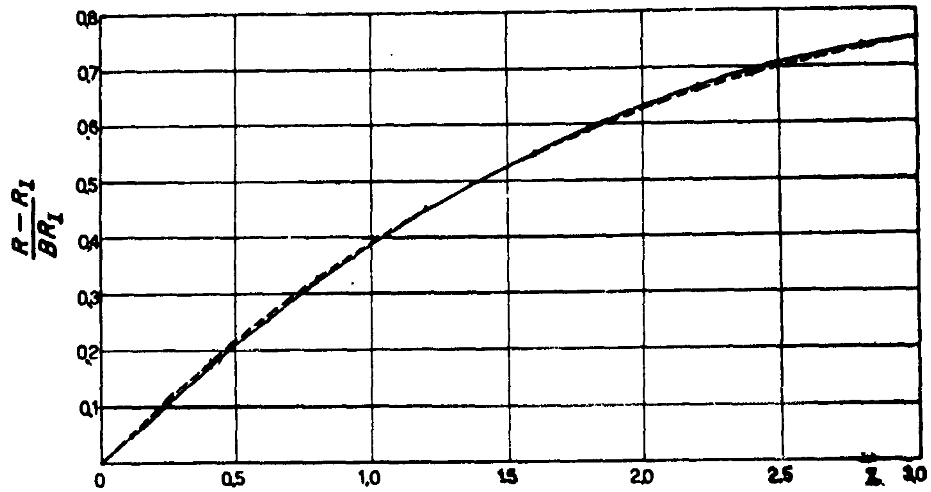

FIGURE 9

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST SHAPE ENCLOSES THE SAME AREA AS THAT COMPRISED BETWEEN THE $N_1 N_2$ CHORDLINE AND THE PARABOLIC CURVE DEFINED ACCORDING TO THE EQUATION:

$$\frac{R}{R_1} = 1 + \beta\,\frac{X}{R_1} + 0.2\,\frac{X}{R_1} - 0.2\,\frac{X^2}{R_1^2}\cdot\frac{R_1}{L}$$

SMALL CONICAL DIFFUSOR ANGLE CASE

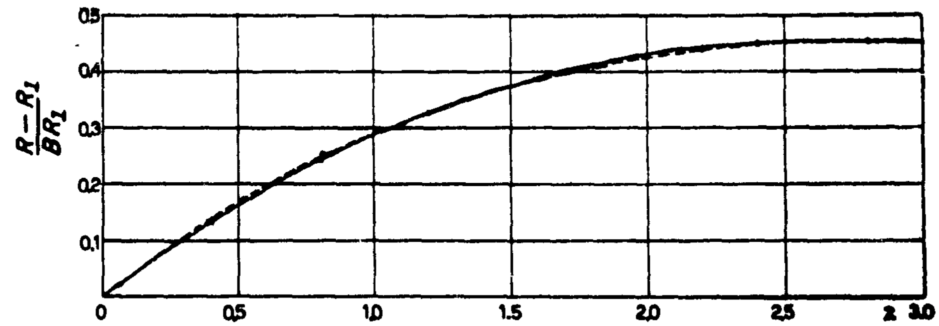

FIGURE 10

LEGEND:

— MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT GEOMETRY BY MEANS OF EQ. (47) OF PART I

--- MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT PLANAR ELEMENTS THROUGH ACKERET'S TWO-DIMENSIONAL FORMULA

DETERMINATION OF THE BEST DUCT CONTOUR UNDER CONDITION (a) TOGETHER WITH CONDITION (d)

8. Calculation of the Coefficients H_n and K_k for Perimetral Condition on Volume

In the previous cases the B and C values which were needed for insertion into Eqs. (65) of Part I, were evaluated by means of Eqs. (64), but now in this instance where it is required to invoke the perimetral condition having to do with the volume enclosed by the duct, as stated formally in the (d) condition of the Introduction, it will be necessary to make use of the following formulations, instead of the ones listed as Eqs. (64) of Part I (the new coefficients will be distinguished from the former ones by affixing an asterisk to the symbol):

$$B_n^* = \int_0^\pi \left[\lambda_1 + (2\lambda_2 \ell_1^* \theta)(1+\beta B \ell_1^* \theta)\right] \cos n\theta \, d\theta$$

$$= -2\ell_1^* \lambda_2 \cdot \frac{2}{n^2} - \beta B \ell_1^{*2} \cdot 2\lambda_2 \cdot \frac{2\pi}{n^2} \qquad \text{for } n \text{ odd}$$

$$= \beta B \ell_1^{*2} \cdot 2\lambda_2 \cdot \frac{2\pi}{n^2} \qquad \text{for } n \text{ even}$$

$$= \pi\lambda_1 + \left(\ell_1^* \pi^2 + \frac{2}{3} B\beta \ell_1^{*2} \pi^3\right)\lambda_2 \qquad \text{for } n = 0$$

Likewise

$$C_n^* = \int_0^\pi \left[\lambda_1 + (2\lambda_2 \ell_1^* \theta)(1+\beta B \ell_1^* \theta)\right] \sin n\theta \, d\theta$$

$$= \frac{2\lambda_1}{n} + \left[2\ell_1^* \frac{\pi}{n} + 2\beta B \ell_1^{*2}\left(\frac{\pi^2}{n} - \frac{4}{n^3}\right)\right] \lambda_2 \qquad \text{for } n \text{ odd}$$

$$= -\left[2\ell_1^* \frac{\pi}{n} + 2\beta B \ell_1^{*2} \cdot \frac{\pi^2}{n}\right] \lambda_2 \qquad \text{for } n \text{ even}$$

$$= 0 \qquad \text{for } n = 0$$

Consequently, by carrying out these computations, for n running up to 6, the following explicit values for the sought coefficients are found:

$$B_o^* = 3.14159\ \lambda_1 + 5.89049\ \lambda_2$$
$$B_1^* = -2.62603\ \lambda_2$$
$$B_2^* = 0.17905\ \lambda_2$$
$$B_3^* = -0.29178\ \lambda_2$$
$$B_4^* = 0.04476\ \lambda_2$$
$$B_5^* = -0.10504\ \lambda_2$$
$$B_6^* = 0.01989\ \lambda_2$$

and

$$C_1^* = 2.00000\ \lambda_1 + 3.66904\ \lambda_2$$
$$C_2^* = -2.06250\ \lambda_2$$
$$C_3^* = 0.66667\ \lambda_1 + 1.35811\ \lambda_2$$
$$C_4^* = -1.03125\ \lambda_2$$
$$C_5^* = 0.40000\ \lambda_1 + 0.82135\ \lambda_2$$
$$C_6^* = -0.68750\ \lambda_2$$

If the same system of notation is employed here as was utilized in the previous Articles, it will be seen that the equations which determine the values of the H^* and H^{**} coefficients, which are obtained by use of the newly determined B and C constants, will take the following explicit concrete form in the present instance

$$0.87169\ H_o^* + 0.00505\ H_2^* + 0.00188\ H_4^* + 0.00100\ H_6^* = 3.61601 + 0.02531\ L_1' + 0.00440\ L_3' + 0.00132\ L_5' + 0.11399\ L_o$$

$$0.00190\ H_o^* + 1.02546\ H_2^* + 0.00131\ H_4^* + 0.00074\ H_6^* = 0.09616 - 0.00843\ L_1' + 0.00793\ L_3' + 0.00157\ L_5' + 0.11399\ L_2$$

$$0.00100\ H_o^* + 0.00181\ H_2^* + 0.99434\ H_4^* + 0.00054\ H_6^* = -0.04987 - 0.00169\ L_1' - 0.00566\ L_3' + 0.00366\ L_5' + 0.11399\ L_4$$

$$0.00063\ H_o^* + 0.00125\ H_2^* + 0.00065\ H_4^* + 0.095638\ H_6^* = -0.03177 - 0.00072\ L_1' - 0.00147\ L_3' - 0.00299\ L_5' + 0.11399\ L_6$$

and likewise

$$0.87169\ H_o^{**} + 0.00505\ H_2^{**} + 0.00188\ H_4^{**} + 0.00100\ H_6^{**} = 6.76705$$
$$+0.02531\ L_1' + 0.00440\ L_3' + 0.00132\ L_5' + 0.11399\ L_o$$
$$0.00190\ H_o^{**} + 1.02546\ H_2^{**} + 0.00131\ H_4^{**} + 0.00074\ H_6^{**} + 0.01325$$
$$-0.00843\ L_1' + 0.00793\ L_3' + 0.00157\ L_5' + 0.11399\ L_2$$
$$0.00100\ H_o^{**} + 0.00181\ H_2^{**} + 0.99434\ H_4^{**} + 0.00054\ H_6^{**} = -0.05063$$
$$-0.00169\ L_1' - 0.00566\ L_3' + 0.00366\ L_5' + 0.11399\ L_4$$
$$0.00063\ H_o^{**} + 0.00125\ H_2^{**} + 0.00065\ H_4^{**} + 0.95638\ H_6^{**} = -0.04244$$
$$-0.00072\ L_1' - 0.00147\ L_3' - 0.00299\ L_5' + 0.11399\ L_6$$

while in like manner, the coefficients with odd subscripts may be determined from

$$0.96562\ H_1^{*} + 0.00171\ H_3^{*} + 0.00060\ H_5^{*} = +0.00901\ L_2' + 0.00251\ L_4'$$
$$+0.00132\ L_6' + 0.11399\ L_1$$
$$0.00295\ H_1^{*} + 1.00167\ H_3^{*} + 0.00045\ H_5^{*} = -0.00540\ L_2' + 0.00537\ L_4'$$
$$+0.00171\ L_6' + 0.11399\ L_3$$
$$0.00197\ H_1^{*} + 0.00086\ H_3^{*} + 0.99258\ H_5^{*} = -0.00129\ L_2' - 0.00418\ L_4'$$
$$+0.00419\ L_6' + 0.11399\ L_5$$

and likewise

$$0.96562\ H_1^{**} + 0.00171\ H_3^{**} + 0.00060\ H_5^{**} = -2.81963 + 0.00901\ L_2'$$
$$+0.00251\ L_4' + 0.00132\ L_6' + 0.11399\ L_1$$
$$0.00295\ H_1^{**} + 1.00167\ H_3^{**} + 0.00045\ H_5^{**} = -0.25289 - 0.00540\ L_2'$$
$$+0.00537\ L_4' + 0.00171\ L_6' + 0.11399\ L_3$$
$$0.00197\ H_1^{**} + 0.00086\ H_3^{**} + 0.99258\ H_5^{**} = -0.06926 - 0.00129\ L_2'$$
$$-0.00418\ L_4' + 0.00419\ L_6' + 0.11399\ L_5$$

Thus, if the L and L' values are guessed to be zero for purposes of making a first approximation, it will be found that the first-stage solutions to these equations will produce the following results for H:

$$H_o^{*(1)} = 4.14828 \quad \text{and} \quad H_1^{*(1)} = H_3^{*(1)} = H_5^{*(1)} = 0$$
$$H_2^{*(1)} = -0.10146$$
$$H_4^{*(1)} = -0.05414$$
$$H_6^{*(1)} = -0.03578$$

and by use of these values it can be determined that the corresponding K values are as follows

$$K_1^{*(1)} = 2.68804 \quad \text{and} \quad K_2^{*(1)} = K_4^{*(1)} = K_6^{*(1)} = 0$$
$$K_3^{*(1)} = 0.80677$$
$$K_5^{*(1)} = 0.45582$$

and consequently the values of L and L' may actually be determined in first approximation by use of Eqs. (66') of Part I, instead of assuming them to be zero as was originally done. The evaluations for the L* and L'* adjusting factors thus turn out to be

$$L_o^{*} = 1.69114 \quad \text{and} \quad L_1'^{*} = 0.74822$$
$$L_1^{*} = -1.39323 \qquad L_2'^{*} = -1.03263$$
$$L_2^{*} = -0.34441 \qquad L_3'^{*} = 0.25727$$
$$L_3^{*} = 0.04695 \qquad L_4'^{*} = -0.38338$$
$$L_4^{*} = -0.17742 \qquad L_5'^{*} = 0.15455$$
$$L_5^{*} = 0.08992 \qquad L_6'^{*} = -0.19400$$
$$L_6^{*} = -0.11272$$

By use of these values of L* and L'*, a better second approximation for the H's and K's can then be obtained, and upon carrying out the arithmetic anew, it is found that

$$H_o^{*(2)} = 4.39371 \quad \text{and} \quad H_1^{*(2)} = -0.17537$$
$$H_2^{*(2)} = -0.14398 \qquad H_3^{*(2)} = -0.00904$$
$$H_4^{*(2)} = -0.07678 \qquad H_5^{*(2)} = 0.01280$$
$$H_6^{*(2)} = -0.05075$$

while

$$K_1^{*(2)} = 2.80785 \qquad \text{and} \qquad K_5^{*(2)} = 0.47491$$
$$K_2^{*(2)} = -0.15688 \qquad\qquad K_6^{*(2)} = -0.02887$$
$$K_3^{*(2)} = 0.84134$$
$$K_4^{*(2)} = -0.05647$$

In an entirely analogous way, the first approximation values for the part of the H and K coefficients that are linked to the λ_2 parameter can be obtained by solution of the equations set down above, and they yield the following specific values at this stage

$$H_0^{**(1)} = 7.76314 \qquad \text{and} \qquad H_1^{**(1)} = -2.92002$$
$$H_2^{**(1)} = -0.00146 \qquad\qquad H_3^{**(1)} = -0.24387$$
$$H_4^{**(1)} = -0.05872 \qquad\qquad H_5^{**(1)} = -0.06377$$
$$H_6^{**(1)} = -0.04945$$

while the corresponding K-values are found to be

$$K_1^{**(1)} = 4.93048 \qquad \text{and} \qquad K_4^{**(1)} = -1.28890$$
$$K_2^{**(1)} = -2.70805 \qquad\qquad K_5^{**(1)} = 0.93674$$
$$K_3^{**(1)} = 1.64579 \qquad\qquad K_6^{**(1)} = -0.84177$$

Recourse can now be had to the same Eqs. (66') of Part I, as employed previously, in order to determine the L^{**} and L'^{**} factors applying to this stage in the computations; the result of such a computation gives

$$L_0^{**} = 3.83467 \qquad \text{and} \qquad L_1'^{**} = 1.86163$$
$$L_1^{**} = -3.33123 \qquad\qquad L_2'^{**} = -2.06417$$
$$L_2^{**} = -0.14754 \qquad\qquad L_3'^{**} = 0.96828$$
$$L_3^{**} = 0.22216 \qquad\qquad L_4'^{**} = -0.78030$$
$$L_4^{**} = -0.39442 \qquad\qquad L_5'^{**} = 0.52456$$
$$L_5^{**} = 0.30960 \qquad\qquad L_6'^{**} = -0.40067$$
$$L_6^{**} = -0.32520$$

and thus upon use of these adjusting factors the second stage approximation for the H^{**} and K^{**} values may be obtained through means of the specific formulae set down above, and it follows that the new values, to better approximation, are now

$$H_0^{**(2)} = 8.32478 \quad \text{and} \quad H_1^{**(2)} = -3.33468$$
$$H_2^{**(2)} = -0.02571 \qquad H_3^{**(2)} = -0.21110$$
$$H_4^{**(2)} = -0.11115 \qquad H_5^{**(2)} = -0.02315$$
$$H_6^{**(2)} = -0.09304$$

and likewise the second step in the approximation to the K-coefficients produces the following recomputed values:

$$K_1^{**(2)} = 5.22260$$
$$K_2^{**(2)} = -3.03105$$
$$K_3^{**(2)} = 1.77885$$
$$K_4^{**(2)} = -1.40629$$
$$K_5^{**(2)} = 1.00331$$
$$K_6^{**(2)} = -0.90232$$

It is considered to be sufficiently accurate to stop the computations at this second stage in the method of successive approximations because it is only the smallest (in absolute value) values of H and K which have shown the greatest relative change in going from the first-step results to the second-step approximations, and thus the ultimate influence of any such small residual error upon the final formula for $\varphi(\theta)$ will be scarcely felt. If one then divides the sought slope function into two parts, as was also done previously, so that

$$\eta(z) = \eta_1(z)\cdot\lambda_1 + \eta_2(z)\,\lambda_2$$

then it follows that the individual parts take on the evaluated forms given below when the appropriate values of H^*, K^*, H^{**}, and K^{**}, just elicited, have been inserted into the general formula for $\eta(z)$, which is the counterpart of

Eq. (62) of Part I that applied in the previous cases. In the present circumstances, therefore, it will be seen that

$$\eta_1(z) = \frac{1}{1+0.25z} + 0.12096$$
$$-0.00191 \cos\theta + 0.00218 \cos 2\theta + 0.00003 \cos 3\theta$$
$$-0.00009 \cos 4\theta + 0.00003 \cos 5\theta - 0.00100 \cos 6\theta$$
$$+0.25142 \sin\theta - 0.00824 \sin 2\theta + 0.04333 \sin 3\theta$$
$$-0.00214 \sin 4\theta + 0.01292 \sin 5\theta - 0.00091 \sin 6\theta$$

while

$$\eta_2(z) = \frac{z(1+0.125z)}{1+0.25z} = 0.22918$$
$$-0.03635 \cos\theta + 0.00039 \cos 2\theta + 0.00066 \cos 3\theta$$
$$-0.00013 \cos 4\theta - 0.00006 \cos 5\theta - 0.00184 \cos 6\theta$$
$$+0.46764 \sin\theta - 0.15926 \sin 2\theta + 0.09161 \sin 3\theta$$
$$-0.05324 \sin 4\theta + 0.02729 \sin 5\theta - 0.02836 \sin 6\theta$$

9. Determination of the λ_1 and λ_2 Parameters for Perimetral Condition on Volume

The stage has now once again been reached in the development of the computations where it is possible to apply the final specific perimetral condition which has to do with the restriction on the volume enclosed by the duct in this instance. The formal statement of this condition is repeated here as:

Case (d) In addition to the closure condition, it is further stipulated now that the volume enclosed between the solid of revolution swept out by the sought contour, when used as a generatrix and revolved about the duct axis, and the frustum of a cone which is marked out when the chordline N_1 and N_2 is used as a generatrix and revolved around the same axis shall be the equal to the volume enclosed between the same inner truncated-cone type of body of revolution and the one which is swept out as a result of a complete rotation, about the duct's central axis, of the parabolic surface defined according to the equation

$$\frac{R}{R_1} = 1+\beta\frac{x}{R_1} + 0.07725\left(\frac{x}{R_1} - \frac{x^2}{R_1^2}\cdot\frac{R_1}{L}\right) \qquad \textbf{(A)}$$

in the case of the Normalizing Mach No., $M_\infty = \sqrt{2}$, or B = 1, and $\beta = 0.25$, while the generating curve is similarly taken to be

$$\frac{R}{R_1} = 1+\beta\frac{x}{R_1} + 0.16058\left(\frac{x}{R_1} - \frac{x^2}{R_1^2}\cdot\frac{R_1}{L}\right) \qquad \text{(B)}$$

in the case of the Small Conical Diffusor Angle, when $M_\infty = 1.9436$, B = 1.6667, and $\beta = 0.15$.

In addition, the parabolic reference meridional line defined in paragraph (c) of the Introduction has also been used in applying this volume-type perimetral condition for both the Normalizing Mach No. Case and the Small Conical Diffusor Angle Case.

In all these instances the first next step is to determine the appropriate values for λ_1 and λ_2. This will be done for each of the four "restricted volume" examples listed above, and they will be treated in the order set down above.

In the Normalizing Mach No. Case, with (A) Type Comparative Surface

It is found that the specific form of the comparative surface being used in this instance will produce the following value for the D constant which appears in Eq. (50) of Part I:

$$D = \frac{L^2}{B^2 R_1^2}\left[\frac{\beta}{2} + \frac{0.07725}{6} + \left(\frac{\beta^2}{3} + \beta\,\frac{0.07725}{12}\right)\frac{L}{R_1}\right] = 1.8468 .$$

The corresponding λ-values may then be determined from the relations

$$3.11405\,\lambda_1 + 5.42245\,\lambda_2 = 0.75$$

$$7.42077\,\lambda_1 + 9.35407\,\lambda_2 = 1.8468$$

which, upon solution, yield the values

$$\lambda = 0.26992 \text{ and } \lambda_2 = -0.01670 .$$

The slope function applying to this example may then be set down as

$$\eta(z) = 0.26992\,\frac{1}{1+0.25z} - 0.01670\,\frac{z\left(1+\frac{0.25z}{2}\right)}{1+0.25z} + 0.02882$$

$$-0.00009 \cos\theta + 0.00058 \cos 2\theta - 0.00002 \cos 3\theta$$
$$-0.00002 \cos 4\theta + 0.00001 \cos 5\theta - 0.00024 \cos 6\theta$$
$$+0.06005 \sin\theta + 0.00044 \sin 2\theta + 0.01017 \sin 3\theta$$
$$+0.00031 \sin 4\theta + 0.00303 \sin 5\theta + 0.00023 \sin 6\theta$$

and, consequently, the expression for the actual radial coordinates comes out to be

$$\begin{aligned}\frac{R-R_1}{BR_1} = {} & 1.21328 \log_e (1+0.25\gamma) - 0.00458\gamma - 0.00417\gamma^2 \\ & +0.00009 \sin\left(\frac{\pi\gamma}{3}\right) + 0.00028 \sin\left(\frac{2\pi\gamma}{3}\right) - 0.00001 \sin(\pi\gamma) \\ & -0.00000 \sin\left(\frac{4\pi\gamma}{3}\right) + 0.00000 \sin\left(\frac{5\pi\gamma}{3}\right) - 0.00004\ (2\pi\gamma) \\ & +0.05734\left[1-\cos\left(\frac{\pi\gamma}{3}\right)\right] - 0.00021\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right] + 0.00324\left[1-\cos(\pi\gamma)\right] \\ & +0.00007\left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right] + 0.00058\left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right] + 0.00004\left[1-\cos(2\pi\gamma)\right]\end{aligned} \tag{17}$$

On the other hand, one may proceed on in a like manner, except for a slight change in the value of D, in order to obtain the ordinates for the

<u>Small Conical Diffuser Angle Case, with (B) Type Comparative Surface</u>

Here the determinative equation giving the sought value of D is

$$D = \frac{L^2}{B^2R_1^2}\left[\frac{\beta}{2} + \frac{0.16058}{6} + \left(\frac{\beta^2}{3} + \beta \cdot \frac{0.16058}{12}\right)\frac{L}{R_1}\right] = 1.3437 .$$

The corresponding values for the λ's are found, therefore, from the equations

$$3.11405\ \lambda_1 + 5.42245\ \lambda_2 = 0.45$$
$$7.42077\ \lambda_1 + 9.35407\ \lambda_2 = 1.3437$$

so that the specific values of λ applying in this example are

$$\lambda_1 = 0.27695 \quad \text{and} \quad \lambda_2 = -0.07606 .$$

Thus one may write down the appropriate slope function almost immediately as

$$\eta(z) = 0.27695\,\frac{1}{1+0.25z} - 0.07606z\,\frac{\left(1+\frac{0.25z}{2}\right)}{1+0.25z} + 0.01607$$

$$+0.00224 \cos\theta +0.00057 \cos 2\theta -0.00006 \cos 3\theta$$
$$-0.00002 \cos 4\theta +0.00001 \cos 5\theta -0.00014 \cos 6\theta$$
$$+0.03406 \sin\theta +0.00983 \sin 2\theta +0.00503 \sin 3\theta$$
$$+0.00346 \sin 4\theta +0.00150 \sin 5\theta +0.00190 \sin 6\theta$$

and it follows, upon integration, that the radial coordinates of the duct contour in this specific case are given by the expression

$$\begin{aligned}\frac{R-R_1}{BR_1} = {} & 1.71628 \log_e (1+0.25\gamma) -0.13605\gamma -0.01901\gamma^2 \\ & +0.00214 \sin\left(\frac{\pi\gamma}{3}\right) +0.00027 \sin\left(\frac{2\pi\gamma}{3}\right) -0.00002 \sin(\pi\gamma) \\ & -0.00000 \sin\left(\frac{4\pi\gamma}{3}\right) +0.00000 \sin\left(\frac{5\pi\gamma}{3}\right) -0.00002 \sin(2\pi\gamma) \\ & +0.03252\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right] +0.00469\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right] +0.00160\left[1-\cos(\pi\gamma)\right] \\ & +0.00083\left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right] +0.00029\left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right] +0.00030\left[1-\cos(2\pi\gamma)\right] \end{aligned} \tag{18}$$

It is again instructive to compare these results with the contour which would result if the two-dimensional approach were to be followed, which relies on use of Ackeret's formula for relating the local pressures to the local duct slopes. Such determinations are easily carried out, following the procedure amply illustrated previously, so that it is at once found that, for

<u>The Normalizing Mach No. Case</u>

The determinative equations for the λ's are this time

$$2.23848\ \lambda_1 + 3.77304\ \lambda_2 = 0.75$$
$$5.46072\ \lambda_1 + 6.09378\ \lambda_2 = 1.8468$$

from whence it follows that

$$\lambda_1 = 0.34435 \qquad \text{and} \qquad \lambda_2 = -0.00552 \quad .$$

The corresponding slope function for the "two-dimensional" case then turns out to be

$$\eta(z) = 0.34435\ \frac{1}{1+0.25z} - 0.00552\ \frac{z\left(1+\frac{0.25z}{2}\right)}{1+0.25z}$$

and thus the sought expression for the radial coordinates of the duct is

$$\frac{R-R_1}{BR_1} = 1.42156 \log_e (1+0.25z) -0.01104z -0.00138z^2 \qquad (17')$$

Likewise

In the Small Conical Diffusor Angle Case

The appropriate determinative equations for the λ's are now

$$2.23848\ \lambda_1 +3.77304\ \lambda_2 = 0.45$$

$$5.46072\ \lambda_1 +6.09378\ \lambda_2 = 1.3437$$

so that thus

$$\lambda_1 = 0.33431 \qquad \text{and} \qquad \lambda_2 = -0.07907.$$

The appropriate slope function is obtained by substitution of these values λ_1 and λ_2 into the usual general expression, with the result this time that one obtains the slope as

$$\eta(z) = 0.33431 \frac{1}{1+0.25z} -0.07907 \frac{z\left(1+\frac{0.25}{2}z\right)}{1+0.25z}$$

with the consequence that the "two-dimensional" duct of best shape comes out to be in this case

$$\frac{R-R_1}{BR_1} = 1.96980 \log_e (1+0.25z)-0.15814z -0.01977\ z^2.$$

The radial coordinates which apply to the ducts in question, which have a minimum drag in the cases where the volume enclosed between the solid of revolution swept out by the sought contour when revolved about the duct's axis and the frustum of a cone which has the chordline $N_1\ N_2$ as a generatrix is equal to the volume enclosed between this same inner truncated-cone type of body of revolution and the one which is swept out as a result of a complete rotation, about the duct's central axis, of the parabolic curve defined according to either of the two equations given in the (d) part of the Introduction, have been calculated by use of Eqs. (17), (18), (17'), and (18') under the specific condition that a normalizing Mach No. is being used in one case, while that a small conical diffusor angle is being assumed in the other, and the results are entered into Table XV below. The results have also been illustrated by means of the plots of Figs. (11) and (12).

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST SHAPE ENCLOSES THE SAME VOLUME AS THAT COMPRISED BETWEEN THE INNER FRUSTUM - OF - A - CONE SURFACE SWEPT OUT BY THE $N_1 N_2$ CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY A PARABOLIC CURVE HAVING THE EQUATION

$$\frac{R}{R_1} = 1 + \beta \frac{x}{R_1} + 0.07725 \left(\frac{x}{R_1} - \frac{x^2}{R_1^2} \cdot \frac{R_1}{L} \right)$$

NORMALIZING MACH NO. CASE

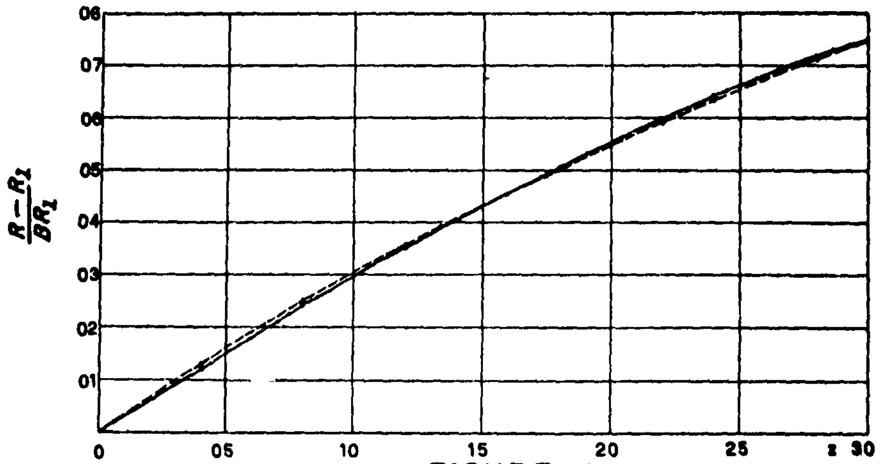

FIGURE 11

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST SHAPE ENCLOSES THE SAME VOLUME AS THAT COMPRISED BETWEEN THE INNER FRUSTUM - OF - A - CONE SURFACE SWEPT OUT BY THE $N_1 N_2$ CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY A PARABOLIC CURVE HAVING THE EQUATION:

$$\frac{R}{R_1} = 1 + \beta \frac{x}{R_1} + 0.16058 \left(\frac{x}{R_1} - \frac{x^2}{R_1^2} \cdot \frac{R_1}{L} \right)$$

—SMALL CONCIAL DIFFUSOR ANGLE CASE

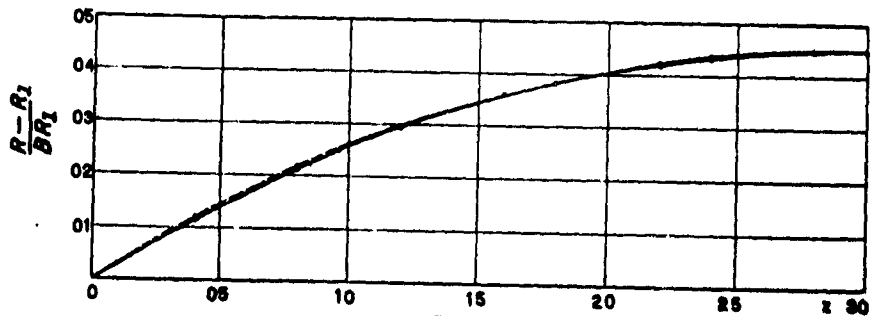

FIGURE 12

LEGEND:

— MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT GEOMETRY BY MEANS OF EQ. (47) OF PART I

--- MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT PLANAR ELEMENTS THROUGH ACKERET'S TWO-DIMENSIONAL FORMULA

TABLE XV

DUCT COORDINATES FOR CASE WHERE SOUGHT BEST DUCT SHAPE ENCLOSES SAME VOLUME AS THAT COMPRISED BETWEEN INNER FRUSTUM OF A CONE SURFACE SWEPTED OUT BY THE N_1N_2 CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY THE ARBITRARILY CHOSEN PARABOLIC CONTOURS DEFINED IN CASE (d) OF INTRODUCTION

γ	Normalizing Mach No. Case $\beta = 1.0$ and $\beta = 0.25$		Small Conical Diffusor Angle Case $\beta = 1.66667$ and $\beta = 0.15$	
	$\frac{R-R_1}{BR_1}$		$\frac{R-R_1}{BR_1}$	
	By Use of Formula (17)	By Use of Formula (17')	By Use of Formula (18)	By Use of Formula (18')
0	0	0	0	0
0.4	0.12163	0.13085	0.11452	0.12132
0.8	0.24135	0.24946	0.21489	0.21997
1.2	0.35297	0.35772	0.29647	0.29856
1.6	0.45723	0.45711	0.36059	0.35914
1.8	0.50631	0.50385	0.38585	0.38319
2.2	0.59822	0.59203	0.42389	0.41967
2.4	0.64130	0.63369	0.43682	0.43239
2.8	0.71845	0.71259	0.44962	0.44745
3.0	0.75000	0.75000	0.45000	0.45000

Since it may possibly be of more advantage, for comparative purposes, to to use the same shape for the "control" surface, regardless of whether the restriction of the perimetral condition has to do with the area enclosed in one meridional plane or the volume enclosed by the revolved curve, the "volume" condition will also now be applied to the case where the comparative contour is given by the parabolic equation set down in the (c) part of the <u>Introduction</u>. Carrying through the same processes, as just illustrated above, for this new case, therefore, it will be necessary first of all to find a new value for D. This will be done for the usual specific instances of the normalizing Mach No. case and the small conical diffusor angle case:

$$D = \frac{L^2}{B^2 R_1^2}\left[\frac{\beta}{2} + \frac{0.2}{6} + \left(\frac{\beta^2}{3} + \frac{0.2}{12}\beta\right)\frac{L}{R_1}\right] = 2.1000 \qquad \text{for } B = 1.0 \text{ and } \beta = 0.25$$

while

$$D = \frac{L^2}{B^2 R_1^2}\left[\frac{\beta}{2} + \frac{0.2}{6} + \left(\frac{\beta^2}{3} + \frac{0.2}{12}\beta\right)\frac{L}{R_1}\right] = 1.4250 \qquad \text{for } B = 1.66667 \text{ and } \beta = 0.15.$$

Consequently the rest of the computation goes through in the same straightforward manner employed previously, so that

In the Normalizing Mach No. Case

The appropriate values of λ will be found from solution of the equations

$$3.11405\ \lambda_1 + 5.42245\ \lambda_2 = 0.7500$$

$$7.42077\ \lambda_1 + 9.35407\ \lambda_2 = 2.1000$$

so that the sought values may be seen to be

$$\lambda_1 = 0.39351 \quad \text{and} \quad \lambda_2 = -0.08768\ .$$

By use of these values of λ in the general expression for the slope function, therefore, it will be seen that in this case

$$\begin{aligned}\eta(z) = {} & 0.39351\ \frac{1}{1+0.25z} - 0.08768\ \frac{z\left(1+\frac{0.25}{2}z\right)}{1+0.25z} + 0.02750 \\ & + 0.00244 \cos\theta + 0.00082 \cos 2\theta - 0.00007 \cos 3\theta \\ & - 0.00003 \cos 4\theta + 0.00002 \cos 5\theta - 0.00023 \cos 6\theta \\ & + 0.05793 \sin\theta + 0.01072 \sin 2\theta + 0.00902 \sin 3\theta \\ & + 0.00383 \sin 4\theta + 0.00269 \sin 5\theta + 0.00213 \sin 6\theta\end{aligned}$$

and then the corresponding radial coordinate function is found to be

$$\begin{aligned}\frac{R-R_1}{BR_1} = {} & 2.27548 \log_e (1+0.25\gamma) - 0.14786\gamma - 0.02192\gamma^2 \\ & + 0.00233 \sin\left(\frac{\pi\gamma}{3}\right) + 0.00039 \sin\left(\frac{2\pi\gamma}{3}\right) - 0.00002 \sin(\pi\gamma) \\ & - 0.00001 \sin\left(\frac{4\pi\gamma}{3}\right) + 0.00000 \sin\left(\frac{5\pi\gamma}{3}\right) - 0.00004 \sin(2\pi\gamma) \\ & + 0.05532\left[1-\cos\left(\frac{\pi\gamma}{3}\right)\right] + 0.00512\left[1-\cos\left(\frac{2\pi\gamma}{3}\right)\right] + 0.00287\left[1-\cos(\pi\gamma)\right] \\ & + 0.00091\left[1-\cos\left(\frac{4\pi\gamma}{3}\right)\right] + 0.00051\left[1-\cos\left(\frac{5\pi\gamma}{3}\right)\right] + 0.00034\left[1-\cos(2\pi\gamma)\right]\end{aligned} \qquad (19)$$

In the Small Conical Diffusor Angle Case

The appropriate values of λ will be obtained from solution of the simultaneous set

$$3.11405\,\lambda_1 + 5.42245\,\lambda_2 = 0.4500$$

$$7.42077\,\lambda_1 + 9.35407\,\lambda_2 = 1.4250$$

and consequently

$$\lambda_1 = 0.31663 \quad \text{and} \quad \lambda_2 = -0.09885$$

with the result that the sought slope function will be given in the form

$$\begin{aligned}\eta(z) = {} & 0.31663 \cdot \frac{1}{1+0.25z} - 0.09885 \cdot \frac{z\left(1+\frac{0.25}{2}z\right)}{1+0.25z} + 0.01564 \\ & +0.00299\cos\theta + 0.00065\cos 2\theta - 0.00007\cos 3\theta \\ & -0.00002\cos 4\theta + 0.00001\cos 5\theta - 0.00013\cos 6\theta \\ & +0.03338\sin\theta + 0.01313\sin 2\theta + 0.00466\sin 3\theta \\ & +0.00458\sin 4\theta + 0.00139\sin 5\theta + 0.00251\sin 6\theta\,.\end{aligned}$$

Upon integration of this function the final equation, describing the way in which the external contour must vary in this instance to give least drag, is obtained, and this computation may be readily performed to give

$$\begin{aligned}\frac{R-R_1}{BR_1} = {} & 2.05732\log_e(1+0.25z) - 0.18206z - 0.02471z^2 \\ & +0.00285\sin\left(\frac{\pi z}{3}\right) + 0.00031\sin\left(\frac{2\pi z}{3}\right) - 0.00002\sin(\pi z) \\ & -0.00000\sin\left(\frac{4\pi z}{3}\right) + 0.00000\sin\left(\frac{5\pi z}{3}\right) - 0.00002\sin(2\pi z) \\ & +0.03188\left[1-\cos\left(\frac{\pi z}{3}\right)\right] + 0.00627\left[1-\cos\left(\frac{2\pi z}{3}\right)\right] + 0.00148\left[1-\cos(\pi z)\right] \\ & +0.00109\left[1-\cos\left(\frac{4\pi z}{3}\right)\right] + 0.00026\left[1-\cos\left(\frac{5\pi z}{3}\right)\right] + 0.00040\left[1-\cos(2\pi z)\right]\end{aligned} \tag{20}$$

By application of the "two-dimensional" treatment in these cases, however, it will be seen that

In the Normalizing Mach No. Case

$$2.23848\,\lambda_1 + 3.77304\,\lambda_2 = 0.75$$

$$5.46072\,\lambda_1 + 6.09378\,\lambda_2 = 2.1006$$

and, hence, one gets that

$$\lambda_1 = 0.48189 \text{ and } \lambda_2 = -0.08712 \quad .$$

The corresponding slope function then takes the form

$$\eta(z) = 0.48189\,\frac{1}{1+0.25z} - 0.08712\,\frac{z(1+\frac{0.25}{2}z)}{1+0.25z}$$

and the related value for the actual radial coordinates then turns out to be

$$\frac{R-R_1}{BR_1} = 2.62452 \log_e (1+0.25z) - 0.17424z - 0.02178z^2 \quad . \tag{19'}$$

Likewise,

In the Small Conical Diffusor Angle Case

One has this time that

$$2.23848\,\lambda_1 + 3.77304\,\lambda_2 = 0.45$$

$$5.46072\,\lambda_1 + 6.09378\,\lambda_2 = 1.4256$$

and thus the appropriate values of λ in this case are

$$\lambda_1 = 0.37869 \quad \text{and} \quad \lambda_2 = -0.10540 \quad .$$

By substitution of these values into the general expression for the slope function, therefore, one finds that now

$$\eta(z) = 0.37869\,\frac{1}{1+0.25z} - 0.10540\,\frac{z(1+\frac{0.25}{2}z)}{1+0.25z}$$

and thus the sought "two-dimensional" solution for the best duct in this case is

$$\frac{R-R_1}{BR_1} = 2.35796 \log_e (1+0.25z) - 0.21080z - 0.02635z^2 \quad . \tag{20'}$$

The radial coordinates which apply to the duct which has a minimum drag under the circumstances that the volume enclosed between the solid of revolution swept out by this best contour when revolved about its axis and the frustum of a cone which has the chordline N_1N_2 as a generatrix is equal to the volume enclosed between this same inner truncated-cone type of body of revolution and the one which is swept out as a result of a complete rotation, about the duct's axis, of the parabolic line defined by means of the equation

$$\frac{R}{R_1} = 1+\beta \frac{x}{R_1} + 0.2 \frac{x}{R_1} - 0.2 \frac{x^2}{R_1^2} \cdot \frac{R_1}{L}$$

in any meridional plane have been computed by use of Eqs. (19), (20),(19'), and (20') under the specific condition that a normalizing Mach No. is governing the general flow as well as in the case that a small conical diffusor angle type of duct is under study. The results have been entered into Table XVI and the results have also been illustrated by means of the plots constituting the appended Figs. (13) and (14).

TABLE XVI

DUCT COORDINATES FOR CASE WHERE SOUGHT BEST DUCT SHAPE IS ONE WHICH ENCLOSES SAME VOLUME AS THAT LEFT BETWEEN INNER FRUSTUM OF A CONE SWEPT OUT BY N_1N_2 CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY THE SAME PARABOLIC CONTOUR USED AS A REFERENCE IN THE AREA-RESTRICTION CASE

γ	Normalizing Mach No. Case B = 1.0 β = 0.25		Small Conical Diffusor Angle Case B = 1.6667 β = 0.15	
	$\frac{R-R_1}{BR_1}$		$\frac{R-R_1}{BR_1}$	
	By Use of Formula (19)	By Use of Formula (19')	By Use of Formula (20)	By Use of Formula (20')
0	0	0	0	0
0.4	0.16625	0.17697	0.12883	0.13620
0.8	0.31665	0.32517	0.23903	0.24440
1.2	0.44397	0.44812	0.32567	0.32773
1.6	0.54989	0.54853	0.39032	0.38865
1.8	0.59444	0.59097	0.41411	0.41131
2.2	0.66797	0.66143	0.44625	0.44208
2.4	0.69717	0.68990	0.45474	0.45054
2.8	0.73830	0.73402	0.45596	0.45438
3.0	0.75000	0.75000	0.45000	0.45000

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST SHAPE ENCLOSES THE SAME VOLUME AS THAT LEFT BETWEEN THE INNER FRUSTUM-OF-A-CONE SWEPT OUT BY THE $N_1 N_2$ CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY THE SAME PARABOLIC CONTOUR USED AS A REFERENCE IN THE AREA-RESTRICTED CASE, AND THE EQUATION OF WHICH IS:

$$\frac{R}{R_1} = 1 + \beta \cdot \frac{x}{R_1} + 0.2\,\frac{x}{R_1} - 0.2\,\frac{x^2}{R_1^2} \cdot \frac{R_1}{L}$$

NORMALIZING MACH NO CASE

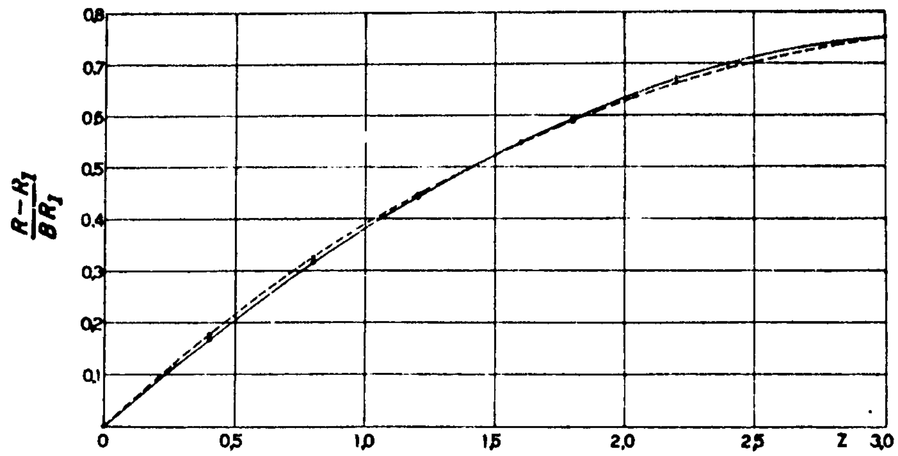

FIGURE 13

DUCT CONTOUR FOR LEAST DRAG IN CASE WHERE THIS BEST SHAPE ENCLOSES THE SAME VOLUME AS THAT LEFT BETWEEN THE INNER FRUSTUM-OF-A-CONE SWEPT OUT BY THE $N_1 N_2$ CHORDLINE AND THE OUTER SURFACE SWEPT OUT BY THE SAME PARABOLIC CONTOUR USED AS A REFERENCE IN THE AREA-RESTRICTION CASE, AND THE EQUATION OF WHICH IS:

$$\frac{R}{R_1} = 1 + \beta \cdot \frac{x}{R_1} + 0.2\,\frac{x}{R_1} - 0.2\,\frac{x^2}{R_1^2} \cdot \frac{R_1}{L}$$

SMALL CONICAL DIFFUSOR ANGLE CASE

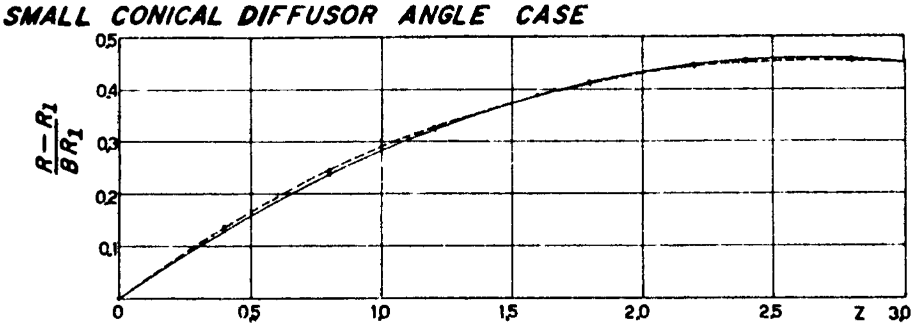

FIGURE 14

LEGEND:

— MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT GEOMETRY BY MEANS OF EQ. (47) OF PART I

--- MERIDIONAL CONTOUR OBTAINED BY ASSUMPTION THAT PRESSURE COEFFICIENTS $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ ARE RELATED TO DUCT PLANAR ELEMENTS THROUGH ACKERET'S TWO-DIMENSIONAL FORMULA

10. Conclusions

By reference to the above-given tabular results for any of the specific sets of circumstances considered, or by visual comparison in the individual figures, it may be seen that in all cases the values of $\frac{R-R_1}{BR_1}$ (the radial coordinates of the sought duct) when calculated through means of the formula which was derived in Part I as Eq. (47), for relating the geometrical features of the duct to the pressures supported by its surface, do not differ in any sizeable degree, to engineering standards of precision, from the corresponding duct coordinates which are obtained by mere application of the "two-dimensional" procedure for determination of the optimal duct shape, which depends on the use of Ackeret's formula for relating the local pressure coefficient, $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$, to the local slopes of the constraining surface.

As has been pointed out in the Introduction, this result should not be interpreted as being of inconsequential import, because it nevertheless, despite the similarity in contours, is going to be found true that the actual values of $\frac{p-p_\infty}{\rho_\infty U_\infty^2}$ that are obtained by use of Eq. (47) of Part I for relating the local planar elements of the ducts in question to the pressures will be appreciably different from the corresponding values for the pressures which are obtained by use of the "two-dimensional" treatment. The extent and import of these differences in pressures, which arise under the dictates of these two contrasting assumptions, may most simply be realized by reference to the plots of the longitudinal components of the induced velocities, produced in these two cases, which are diagrammed in Figs. 3 and 4 for each of the harmonic components making up the total induced velocity; it will be seen that for the lower values of n and k there are marked differences in the plotted curves for the longitudinal velocity components.

On the other hand, if one is merely interested in the determination of the ordinates of the best duct, without concern about how to compute the corresponding pressures, it is obvious that this general result, adduced from the comparisons now carried out and on the basis of the assurances given in the text about the wide generality of the illustrative examples selected for computation, may be of quite wide practical interest, because the solution of the problem, as far as determining the optimal duct ordinates is concerned, may now be attained by use of the simplest of procedures, depending merely upon application of Ackeret's two-dimensional formula. Of course, it will still be necessary to use the more complicated formulae developed in Part I for relating the duct contours to the pressures exerted on its external walls if one wishes to obtain suitably exact values for the pressures.

REFERENCE

1. Ferrari, C. Determination of the External Contour of a Body of Revolution with a Central Duct so as to Give Minimum Drag in Supersonic Flow, with Various Perimetral Conditions Imposed upon the Missile Geometry, Parts I and II, Cornell Aeronautical Laboratory Report No. AF-814-A-1, Contract No. N6ori-119, T.O.IV, March 1953.